ADVANCED EXAMINATION □ ALGEBRA □ September 28, 1996

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Name: ____

#	#	#	#	#	#	#	#	total (160)

Work any eight problems. Indicate which problems you are doing in the top parts of the boxes above.

1. For each of the following groups find all subgroups of the given group and sketch the resulting lattice.

   (a) $(\mathbf{Z}_6, +)$ (b) $(\mathbf{Z}_4 \times \mathbf{Z}_4, +)$ (c) S_3

2. Suppose G and H are groups and $f\colon G \to H$ is a group homomorphism. Prove that

   (a) $\ker f \lhd G$ (b) $f(G) < H$

3. Let G be the group of rotations of the plane around the origin under composition. Define $f\colon (\mathbf{R}, +) \to G$ by letting $f(t)$ be the clockwise rotation by angle $2\pi t$.

   (a) Prove that f is a group homomorphism.
   (b) What subgroup of $\mathbf{R}$ is the kernel of f?
   (c) Prove that $G \cong \mathbf{R}/\mathbf{Z}$.

4. Suppose the order of a group G is a prime number p. Prove that $G \cong \mathbf{Z}_p$.

5. Let $n \geq 3$ and $1 \leq k \leq n$. Let $K = \{\sigma \in S_n \colon \sigma(k) = k\}$. Prove that

   (a) $K < S_n$ (b) $K \ntriangleleft S_n$

6. Prove that $\mathbf{Z}$ is the free group generated by $\{1\}$.

7. Prove that an ideal of $\mathbf{Z}$ is a prime ideal if and only if it is generated by a prime number.

8. Suppose R is a commutative ring with 1 and I is a maximal ideal of R. Prove that I is a prime ideal of R.

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9. Suppose R is an integral domain.

   (a) Show that if P is a prime ideal of R, then its complement $R \setminus P$ is closed under multiplication.

   (b) Show that if R has exactly one maximal ideal M, then $R \setminus M$ is the multiplicative group of units of R.

   (c) Show that R is a field $\Leftrightarrow$ R has no proper nonzero ideals.

10. Let R be the ring of all continuous real valued functions of a real variable, i.e. $R = \{f\colon \mathbf{R} \to \mathbf{R}\colon f \text{ is continuous}\}$, where addition and multiplication of functions are defined pointwise, i.e. $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$.

    Given a subset of the real line $V \subseteq \mathbf{R}$ define $I(V)$ to be the set of all continuous functions that vanish on V, i.e. $I(V) = \{f \in R\colon \forall x \in V\ f(x) = 0\}$.

    (a) Show that if $V \subseteq \mathbf{R}$, then $I(V)$ is an ideal of R.
    (b) Show that if $a \in \mathbf{R}$, then $I(\{a\})$ is a prime ideal of R.
    (c) Show that if $a, b \in \mathbf{R}$ and $a \neq b$, then $I(\{a, b\})$ is not a prime ideal of R.

11. Prove that

    (a) Every finite integral domain is a field.
    (b) If R is an integral domain and $S = R \setminus \{0\}$, then $S^{-1}R$ is a field.