## Vector area element

$$d\bar{S} = \begin{bmatrix} dy\, dz \\ dz\, dx \\ dx\, dy \end{bmatrix}$$

$$d\bar{S} = \hat{n}\, \underbrace{|d\bar{S}|}_{\text{a.k.a. } d\sigma}$$

**Plane:** $ax + by + z = d$ (make $c=1$ first)

$$\Phi(x,y) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ d - ax - by \end{bmatrix} \qquad d\Phi = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} dx \\ dy \\ -a\,dx - b\,dy \end{bmatrix}$$

$$d\bar{S} = \begin{bmatrix} dy\, dz \\ dz\, dx \\ dx\, dy \end{bmatrix} = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} dx\,dy = \underbrace{\frac{1}{\sqrt{a^2+b^2+1}} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}}_{\hat{n}} \underbrace{\sqrt{a^2+b^2+1}\, dx\,dy}_{|dS|}$$

**Cylinder:** $x^2 + y^2 = a^2$

$$\Phi(\theta,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a\cos\theta \\ a\sin\theta \\ z \end{bmatrix} \qquad d\Phi = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} -a\sin\theta\, d\theta \\ a\cos\theta\, d\theta \\ dz \end{bmatrix}$$

$$d\bar{S} = \begin{bmatrix} dy\, dz \\ dz\, dx \\ dx\, dy \end{bmatrix} = \begin{bmatrix} a\cos\theta \\ a\sin\theta \\ 0 \end{bmatrix} d\theta\, dz = \underbrace{\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}}_{\hat{n} = \hat{r}} \underbrace{a\, d\theta\, dz}_{|d\bar{S}|}$$

**Disc:** $x^2 + y^2 \le a$, $z = c$

$$\Phi(r,\theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ c \end{bmatrix} \qquad d\Phi = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} dr\cos\theta - r\sin\theta\, d\theta \\ dr\sin\theta + r\cos\theta\, d\theta \\ 0 \end{bmatrix}$$

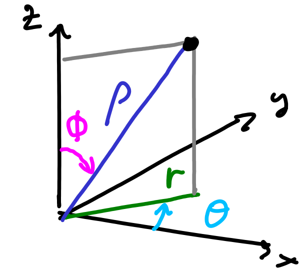
$$d\bar{S} = \begin{bmatrix} dy\, dz \\ dz\, dx \\ dx\, dy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} dr\, d\theta = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\hat{n} = \hat{k}} \underbrace{r\, dr\, d\theta}_{|d\bar{S}|}$$

## Spherical coordinates

Latitude: $-\frac{\pi}{2} \le \lambda \le \frac{\pi}{2}$

$\phi = \frac{\pi}{2} - \lambda$: $\quad 0 \le \phi \le \pi$

Longitude: $-\pi \le \theta \le \pi$




$r = \rho \sin\phi$

$x = r\cos\theta = \rho \sin\phi \cos\theta$

$y = r \sin\theta = \rho\sin\phi \sin\theta$

$z = \rho\cos\phi$

Sphere: $x^2 + y^2 + z^2 = a^2$

$$\Phi(\phi,\theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a\sin\phi\cos\theta \\ a\sin\phi\sin\theta \\ a\cos\phi \end{bmatrix}$$

$$d\Phi = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} a\cos\phi\, d\phi\cos\theta - a\sin\phi\sin\theta\, d\theta \\ a\cos\phi\, d\phi\, \sin\theta + a\sin\phi\cos\theta\, d\theta \\ -a\sin\phi\, d\phi \end{bmatrix}$$

$$dS = \begin{bmatrix} dy\, dz \\ dz\, dx \\ dx\, dy \end{bmatrix} = \begin{bmatrix} a^2(\sin\phi)^2\cos\theta \\ a^2(\sin\phi)^2\sin\theta \\ a^2\cos\phi\sin\phi(\cos\theta)^2 + a^2\sin\phi\cos\phi(\sin\theta)^2 \end{bmatrix} d\phi\, d\theta =$$

$$= \underbrace{\begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix}}_{\hat{n}} \underbrace{a^2 \sin\phi\, d\phi\, d\theta}_{|d\overline{S}|}$$

$\hat{n} = \hat{\rho}$