Energy, entropy and uniqueness

Wave equation in one spatial dimension: $u_{tt} = c^2 u_{xx}$

Boundary conditions: $u(0,t) = u(L,t) = 0$

Initial conditions: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

Solution by separation of variables and Fourier series: $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[B_n \cos\left(\frac{cn\pi}{L}t\right) + D_n \sin\left(\frac{cn\pi}{L}t\right)\right]$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$ and $D_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$

Energy density: $\varepsilon(x,t) = \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right)$
You can think of the first term as kinetic energy density and the second as potential energy density.

Total energy: $E(t) = \int_0^L \varepsilon(x,t)\,dx$

Conservation of total energy: By the product rule $(u_t u_x)_x = u_{tx}u_x + u_t u_{xx}$
Thus $\varepsilon_t = u_t u_{tt} + c^2 u_x u_{xt} = u_t u_{tt} + c^2\left[(u_t u_x)_x - u_t u_{xx}\right] = u_t(u_{tt} - c^2 u_{xx}) + c^2 (u_t u_x)_x = c^2(u_t u_x)_x$
$E_t = \int_0^L \varepsilon_t\,dx = c^2\int_0^L (u_t u_x)_x\,dx = c^2\; u_t u_x\Big|_{x=0}^{x=L} = c^2\left[u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)\right]$
By the boundary conditions $u_t(0,t) = u_t(L,t) = 0$, so $E_t(t) = 0$, so $E(t)$ is constant.

Uniqueness:
Given two solutions satisfying the same initial conditions, their difference is a solution u satisfying $u(x,0) = 0$ and $u_t(x,0) = 0$.
In this case $u_x(x,0) = 0$, so $\varepsilon(x,0) = 0$, so $E(0) = 0$, and by the conservation of energy $E(t) = 0$.
Since $\varepsilon(x,t) \geq 0$, we have $\varepsilon(x,t) = 0$, so $u_x(x,t) = u_t(x,t) = 0$, so $u(x,t)$ is constant, and since $u(x,0) = 0$, $u(x,t) = 0$.

Heat equation in three spatial dimensions: $u_{tt} = c^2\nabla^2 u$

By Duhamel's priniciple, the total heat in a small volume Ω is $Q \approx M\,u\,\mathrm{vol}(\Omega)$, where M is specific heat of matter.

By Newton's law of cooling, heat flux across the boundary $\partial\Omega$ is proportional to temperature gradient: $Q_t = N\int_{\partial\Omega}\nabla u\cdot\widehat{n}\,dS$

By the Gauss-Ostrogradski divergence theorem $Q_t = N\int_{\Omega}\nabla\cdot\nabla u\,dV \approx N\,\nabla^2 u\,\mathrm{vol}(\Omega)$

Dividing $M\,u_t\,\mathrm{vol}(\Omega) \approx N\,\nabla^2 u\,\mathrm{vol}(\Omega)$ by the volume and taking limit as $\mathrm{vol}(\Omega) \to 0$ we obtain $u_t = \frac{N}{M}\nabla^2 u$

Entropy: Define entropy density $\varepsilon = \frac{1}{2}u^2$ and integrate over Ω: $E(t) = \int_{\Omega}\varepsilon\,dV = \frac{1}{2}\int_{\Omega}u^2\,dV$

Entropy principle: In the presence of temperature gradients, total entropy of an insulated body decreases.

Product rule for divergence: $\nabla\cdot(\varphi\Phi) = \nabla\varphi\cdot\Phi + \varphi(\nabla\cdot\Phi)$ implies $\nabla\cdot(u\nabla u) = \nabla u\cdot\nabla u + u(\nabla^2 u)$.
Thus, $\varepsilon_t = u\,u_t = u\,c^2\nabla^2 u = c^2\left[\nabla\cdot(u\nabla u) - (\nabla u)\cdot(\nabla u)\right]$.
Integrating and applying the divergence theorem we obtain $E_t = c^2\left[\int_{\partial\Omega}u\nabla u\cdot\widehat{n}\,dS - \int_{\Omega}(\nabla u)\cdot(\nabla u)\,dV\right]$.

For an insulated body $\nabla u\cdot\widehat{n} = 0$ on the boundary $\partial\Omega$, so $E_t = -c^2\int_{\Omega}(\nabla u)\cdot(\nabla u)\,dV \leq 0$.

Uniqueness: Given two solutions with the same initial state, their difference u is a solution with initial state 0.
Its initial entropy is 0. Since $E \geq 0$ and cannot increase ($E_t \leq 0$), it stays 0.
Therefore, at any time, $\nabla u = 0$, so u is a constant and thus $u = 0$.

Heat equation in one spatial dimension: $u_{tt} = c^2 u_{xx}$

Boundary conditions: $u(0,t) = u(L,t) = 0$

Initial condition: $u(x,0) = f(x)$

Fourier series solution: $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\exp\left(-\left[\frac{cn\pi}{L}\right]^2 t\right)$, where $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$