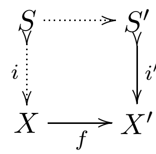
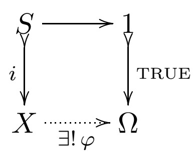
Subobject functor: Suppose $\mathcal{C}$ is a small category. Given $X \in \mathrm{Obj}(\mathcal{C})$, a subobject is an equivalence class of monomorphisms $i: S \to X$, where $i \sim i'$ whenever there exists an isomorphism $\varphi: S \to S'$ such that $i = i'\varphi$. Let $\mathrm{Sub}\, X$ be the set of all subobjects of X (to ensure that this is a set we assume that $\mathcal{C}$ is well-powered). Given a morphism $f: X \to X'$ define $\mathrm{Sub}\, f$ via pullback: $i': S' \to X'$ maps to $i: S \to X$ making the diagram



cartesian. Sub is a presheaf — a contravariant functor from $\mathcal{C}$ to Sets, i.e. $\mathrm{Sub} \in \mathrm{Obj}(\mathrm{Sets}^{\mathcal{C}^{\mathrm{op}}})$.

Subobject classifier: Suppose $\mathcal{C}$ is a category with finite limits. A subobject classifier represents Sub, i.e. it is a monomorphism $\text{TRUE}: 1 \to \Omega$ satisfying a universal property: given any monomorphism $i: S \to X$, there exists unique morphism $\varphi: X \to \Omega$ such that the diagram



is cartesian (a pullback).

- $\mathrm{Sub}\,- \cong \mathcal{C}[-, \Omega]$
- Like any universal construction, subobject classifier is unique up to isomorphism.
- In Sets $1 = \{0\}$, $\Omega = 2 = \{0, 1\}$, and $\text{TRUE}: 1 \to 2$ maps $0 \mapsto 0$.

The category of actions: Suppose M is a monoid. Given a set X, an M-action on X is a function $X \times M \xrightarrow{\cdot} X$ such that $x \cdot e = x$ and $x \cdot (m_1 m_2) = (x \cdot m_1) \cdot m_2$. A morphism of two M-actions is a function $f: X \to X'$ such that $f(x \cdot m) = f(x) \cdot m$.

In this category (BM)

- the terminal object is the trivial action $1 \times M \to 1$,
- A subobject of an M-action $X \times M \to X$ is the inherited action $S \times M \to S$, where S is a subset of X that is closed under the action.
- the subobject classifier is an M-action $\Omega \times M \to \Omega$, where Ω is the set of all right ideals of M and $I \cdot m = \{k \in M : mk \in I\}$,
- $\text{TRUE}: 1 \to \Omega$ maps $0 \mapsto M$.
- Given a subset $S \to X$ closed under the action, define $\varphi: X \to \Omega$ by $x \mapsto \{k \in M : x \cdot k \in S\}$. Note that $x \in S \Leftrightarrow e \in \varphi(x) \Leftrightarrow \varphi(x) = M$, i.e. $S = \varphi^{-1}(\{M\})$.

If M is a group, $\Omega = 2 = \{\{e\}, M\}$ with the trivial action.