Stereographic Projection, the Riemann Sphere, and the Chordal Metric.

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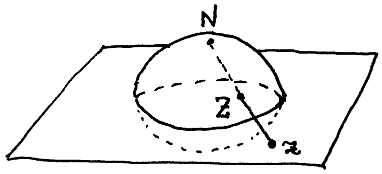

Fig. 1

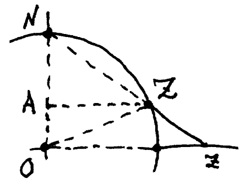

Fig. 2

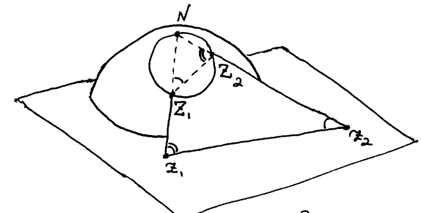

Fig. 3

Definition: The Riemann sphere is the unit sphere $\mathbf{S} = \{Z \in \mathbf{R}^3\colon\ |Z| = 1\}$ and we use the x-y plane to represent $\mathbf{C}$. Each point $z \in \mathbf{C}$ corresponds to a point $Z \in \mathbf{S}$ by *stereographic projection* to the north pole N (Fig. 1).

Proposition 1: A point $z = x + iy \in \mathbf{C}$ is stereographically projected to a point $Z = (\xi, \eta, \zeta) \in \mathbf{S}$, where

$$\xi = \frac{2x}{x^2+y^2+1}, \qquad \eta = \frac{2y}{x^2+y^2+1}, \qquad \zeta = \frac{x^2+y^2-1}{x^2+y^2+1}; \qquad z = \frac{\xi + i\eta}{1-\zeta}.$$

Proof: The segment Nz is given by $\{r(t) = (tx, ty, 1-t)\colon 0 \le t \le 1\}$. Now $r(t) \in \mathbf{S}$ when $|r(t)|^2 = 1$. This means we need to solve $t^2\,|z|^2 + (1-t)^2 = 1$ for t. This is a quadratic equation $t^2(|z|^2+1) - 2t = 0$ with two roots $t = 0$ and $t = \dfrac{2}{|z|^2+1}$. Thus, $N = r(0)$ and $Z = r\left(\dfrac{2}{|z|^2+1}\right) = \left(\dfrac{2x}{|z|^2+1}, \dfrac{2y}{|z|^2+1}, 1 - \dfrac{2}{|z|^2+1}\right)$, which gives the first three formulas. The last formula follows from the similarity of triangles NOz and NAZ (Fig. 2). ∎

Proposition 2: Suppose $z_1, z_2 \in \mathbf{C}$ correspond to $Z_1, Z_2 \in \mathbf{S}$. Then $|Z_1 - Z_2| = \dfrac{2\,|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$.

Proof: Let $z = z_1$ or z_2. Similarity of triangles NOz and NAZ (Fig. 2) implies $\dfrac{|N-z|}{1} = \dfrac{|N-Z|}{1-\zeta}$. But $|N - z| = \sqrt{|z|^2+1}$ and $1 - \zeta = \dfrac{2}{|z|^2+1}$, so $|N - Z| = \dfrac{2}{\sqrt{1+|z|^2}}$ and $|N-z|\,|N-Z| = 2$. The plane Nz_1z_2 intersects $\mathbf{S}$ in a circle (Fig. 3). Since $|N-z_1|\,|N-Z_1| = |N-z_2|\,|N-Z_2| = 2$, the triangles Nz_1z_2 and NZ_2Z_1 are similar, so $\dfrac{|Z_1-Z_2|}{|z_1-z_2|} = \dfrac{|N-Z_2|}{|N-z_1|}$. ∎

Proposition 3: Let $\chi(z_1, z_2) = |Z_1 - Z_2| = \dfrac{2\,|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$. Then χ is a metric on $\mathbf{C}$ (called the *chordal metric*), i.e. (a) $\chi(z_1,z_2) = \chi(z_2,z_1)$, (b) $\chi(z_1,z_2) \ge 0$ and $\chi(z_1,z_2) = 0 \Rightarrow z_1 = z_2$, (c) $\chi(z_1,z_3) \le \chi(z_1,z_2) + \chi(z_2,z_3)$. (Hille, p. 43)

Theorem 1: Stereographic projection is conformal (angle preserving).

Proof 1: Let $a(Z_1, Z_2)$ denote arclenth from Z_1 to Z_2 along the circle that is the intersection of $\mathbf{S}$ with the plane NZ_1Z_2 (Fig. 3). If α is $\angle Z_1NZ_2$, then $\dfrac{a(Z_1,Z_2)}{|Z_1-Z_2|} = \dfrac{\alpha}{\sin(\alpha)} \to 1$ as $\alpha \to 0$. Thus, as $z_2 \to z_1$, $\dfrac{a(Z_1,Z_2)}{|z_1-z_2|} \to \dfrac{1}{|z_1|^2+1}$. Since the magnification factor depends only on z_1 we are done. ∎

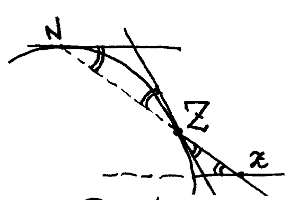

Fig. 4

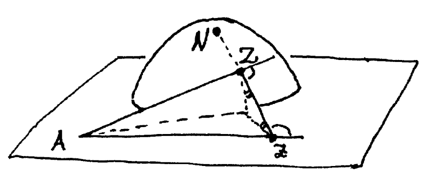

Fig. 5

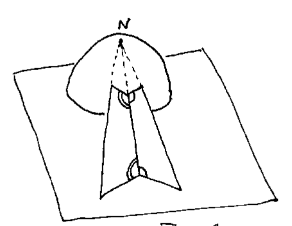

Fig. 6

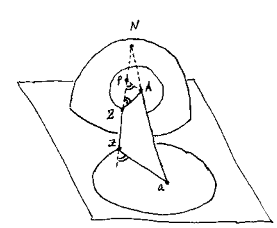

Fig. 7

Proof 2: The line Nz makes equal angles with $\mathbf{C}$ and the tangent plane to $\mathbf{S}$ at Z (Fig. 4). Pick a line AZ tangent to $\mathbf{S}$ at Z. If this line is not parallel to $\mathbf{C}$ we may choose $A \in \mathbf{C}$. Then AZ and Az make equal angles with Nz (Fig. 5). If Az is parallel to $\mathbf{C}$ then these angles are still equal (both $\pi/2$). Given two lines tangent to $\mathbf{S}$ at Z, each line and its image in $\mathbf{C}$ make equal angles with Nz, so the angles between the two tangent lines and their images are equal as well (Fig. 6).

Theorem 2: Stereographic projection is circle preserving.

Proof: Pick a circle on $\mathbf{S}$ not containing N and let A be the vertex of the cone tangent to $\mathbf{S}$ at this circle (Fig. 7). In the plane NZA construct Ap parallel to az. As Z traverses the circle, $|A - Z|$ is constant, but Az and az make equal angles with Nz, so the triangle AZp is isoceles and $|A - Z| = |A - p|$. Furthermore, the triangles NAp and Naz are similar, so $\dfrac{|a-z|}{|A-p|} = \dfrac{|N-a|}{|N-A|}$ which stays constant. Thus, $|a - z|$ is constant. If the circle on $\mathbf{S}$ contains N, we get a line (Fig. 3). ∎