Higher derivatives as multilinear maps

Suppose $U \subseteq \mathbf{R}^n$ is a domain (i.e. open and connected) and $f : U \to \mathbf{R}^m$ is differentiable at all $x \in U$. As we have seen, the derivative map of f at x is a linear map of Δx, i.e. $f'(x) \colon \mathbf{R}^n \to \mathbf{R}^m$. Making the dependence on x explicit, we get a linear-map-valued function of x, i.e. $f' \colon U \to \mathrm{Hom}\,[\mathbf{R}^n, \mathbf{R}^m]$. The derivative map of the first derivative is a linear map of say $\overline{\Delta x}$, i.e. $f''(x) \colon \mathbf{R}^n \to \mathrm{Hom}\,[\mathbf{R}^n, \mathbf{R}^m]$. We may think of this as a bilinear map of Δx and $\overline{\Delta x}$, i.e. $f''(x) \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^m$. Similarly we see that the k-th derivative is a k-linear map $f^{(k)}(x) \colon (\mathbf{R}^n)^k \to \mathbf{R}^m$. It turns out that these multilinear maps are symmetric.

Cartesian representation: Writing $\Delta x = (\Delta x_1, ...\Delta x_n)$ we may represent the linear map $f'(x)$ by multiplication with the derivative matrix (the matrix of partial derivatives, a.k.a. the Jacobian matrix)

$$f'(x)(\Delta x) = \sum_{i=1}^{n} f_i(x)\Delta x_i$$

Second derivative: For simplicity assume that $n = 2$ and $m = 1$. Taking the derivative we obtain

$$\begin{aligned} f''(x)(\Delta x, \overline{\Delta x}) &= (f'(x)(\Delta x))(\overline{\Delta x}) = f'(x)(\Delta x)_1\overline{\Delta x}_1 + f'(x)(\Delta x)_2\overline{\Delta x}_2 \\ &= (f_1(x)\Delta x_1 + f_2(x)\Delta x_2)_1\overline{\Delta x}_1 + (f_1(x)\Delta x_1 + f_2(x)\Delta x_2)\overline{\Delta x}_2 \\ &= f_{11}(x)\Delta x_1\overline{\Delta x}_1 + f_{21}(x)\Delta x_2\overline{\Delta x}_1 + f_{12}(x)\Delta x_1\overline{\Delta x}_2 + f_{22}(x)\Delta x_2\overline{\Delta x}_2 \end{aligned}$$

The matrix representing this bilinear map is the matrix of second partial derivatives of f. Note that this is a symmetric matrix.

$$f''(x)(\Delta x, \overline{\Delta x}) = (\Delta x_1, \Delta x_2)\begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{pmatrix}\begin{pmatrix} \overline{\Delta x}_1 \\ \overline{\Delta x}_2 \end{pmatrix}$$

More generally with arbitrary n and m we can write

$$f''(x)(\Delta x, \overline{\Delta x}) = \sum_{j=1}^{n} f'(x)(\Delta x)_j\overline{\Delta x}_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f_i(x)\Delta x_i\right)_j\overline{\Delta x}_j = \sum_{i,j=1}^{n} f_{ij}(x)\Delta x_i\overline{\Delta x}_j$$

Higher derivatives: We can obtain similar formulas for higher derivatives with the symmetric tensor of higher partial derivatives.

Taylor polynomials: Suppose f is k times differentiable at x. Then $f(x+\Delta x)$ is "approximated" (as $\Delta x \to 0$) by the k-th Taylor polynomial

$$f(x+\Delta x) \approx f(x) + f'(x)(\Delta x) + \frac{1}{2}f''(x)(\Delta x, \Delta x) + \frac{1}{6}f'''(x)(\Delta x, \Delta x, \Delta x) + ...\frac{1}{k!}f^{(k)}(x)(\Delta x, \Delta x, ...\Delta x)$$

For example in cartesian coordinates with $n = 2$ and $m = 1$ we get

$$\begin{aligned} f(x_1+\Delta x_1, x_2+\Delta x_2) &\approx f(x) + f_1(x)\Delta x_1 + f_2(x)\Delta x_2 + \frac{1}{2}f_{11}(x)\Delta x_1^2 + f_{12}(x)\Delta x_1\Delta x_2 + \frac{1}{2}f_{22}(x)\Delta x_2^2 \\ &+ ...\frac{1}{k!}\sum_{i_1,i_2,...i_k=1}^{n} f_{i_1 i_2 ... i_n}(x)\Delta x_{i_1}\Delta x_{i_2}...\Delta x_{i_k} \end{aligned}$$