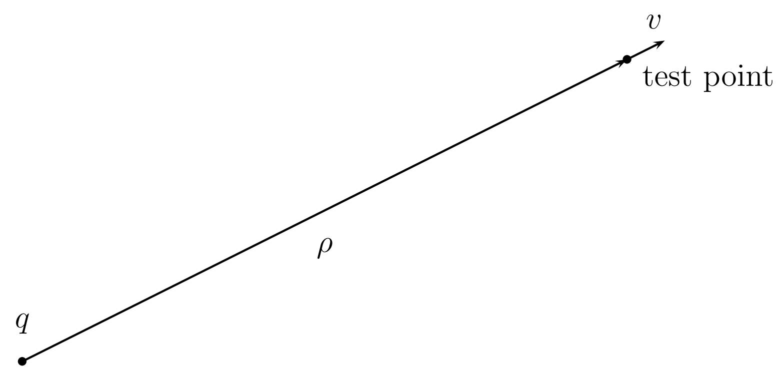
Velocity field induced by a source

(Gravity: Newton, 1679; Electric field due to charge: Charles Coulomb 1785)



A source q induces velocity v at a test point with displacement vector $\rho = (x, y, z)$:

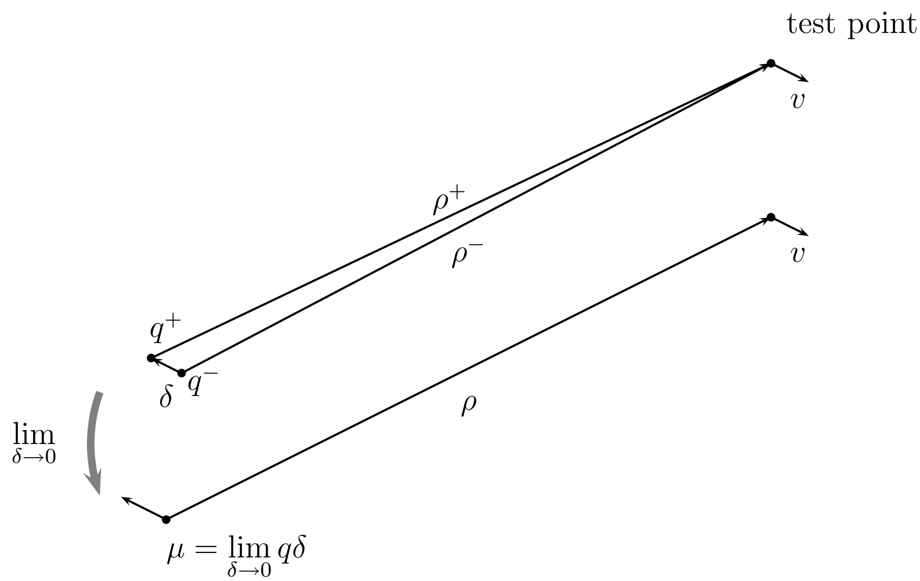
$$v = \frac{q}{4\pi} \frac{\rho}{|\rho|^3} = \frac{q}{4\pi} \frac{\widehat{\rho}}{|\rho|^2}$$

Theorem:

(i) [Lagrange, 1773] $v = \nabla\varphi$, where $\varphi = -\frac{q}{4\pi} \frac{1}{|\rho|}$

(ii) Except at the source q, v is solenoidal ($\nabla \cdot v = 0$), so φ is harmonic.

Velocity field induced by a doublet



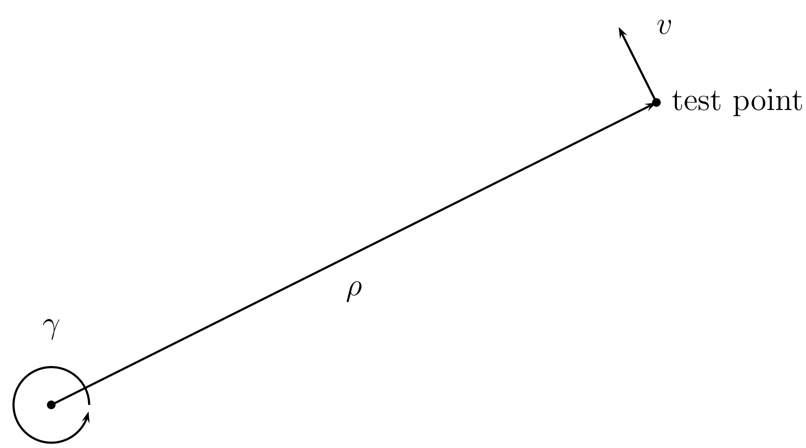
$$\varphi = \lim_{\delta\to 0}\left[-\frac{q}{4\pi}\frac{1}{|\rho^+|} + \frac{q}{4\pi}\frac{1}{|\rho^-|}\right] = -\frac{1}{4\pi}\lim_{\delta\to 0} q\,|\delta|\left[\frac{1}{|\rho^+|} - \frac{1}{|\rho^-|}\right]/|\delta| = -\frac{|\mu|}{4\pi}\frac{\partial}{\partial\hat{\mu}}\frac{1}{|\rho|}$$

Theorem: Except at the doublet, φ is harmonic.

Proof: Superposition.

Velocity field induced by a vortex

(Magnetism: Jean-Baptiste Biot, Felix Savart, 1820)



A vortex γ induces velocity v at a test point with displacement vector $\rho = (x, y, z)$:

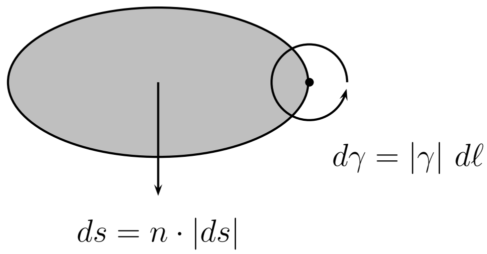
$$v = \frac{\gamma}{4\pi} \times \frac{\rho}{|\rho|^3} = \frac{\gamma}{4\pi} \times \frac{\widehat{\rho}}{|\rho|^2}$$

Theorem: v is solenoidal ($\nabla \cdot v = 0$)

Proof:

$$\nabla \cdot v = \frac{1}{4\pi} \nabla \cdot \frac{\gamma \times \rho}{|\rho|^3} = \frac{1}{4\pi} \left[(\nabla \times \gamma) \cdot \frac{\rho}{|\rho|^3} - \gamma \cdot \left(\nabla \times \frac{\rho}{|\rho|^3} \right) \right] = \frac{1}{4\pi} \gamma \cdot \left[\nabla \times \left(\nabla \frac{1}{|\rho|} \right) \right] = 0$$

Velocity field induced by a vortex filament



Velocity induced by a filament is a line integral $v = \frac{1}{4\pi}\int_{\partial\mathscr{D}} d\gamma \times \nabla\frac{1}{|\rho|}$

Theorem:

(i) $v = \nabla\varphi$, where $\varphi = \frac{|\gamma|}{4\pi}\int_{\mathscr{D}} \nabla\frac{1}{|\rho|}\cdot ds$

(equivalent to a uniform doublet distribution on $\mathscr{D}$)

(ii) v is solenoidal ($\nabla \cdot v = 0$), so φ is harmonic.

Proof: For any vector field F, we have $\int_{\partial\mathscr{D}} F \times d\ell = \int_{\mathscr{D}} \nabla \cdot F\, ds - \nabla \int_{\mathscr{D}} F \cdot ds$

Therefore, since $\frac{1}{|\rho|}$ is harmonic, $v = \frac{|\gamma|}{4\pi}\nabla\int_{\mathscr{D}}\nabla\frac{1}{|\rho|}\cdot ds$