Poisson summation formula

$$\sum_{k=-\infty}^{\infty} g(x+2\pi k) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \widehat{g}(n)e^{inx}$$

Proof: Let $h(x) = \sum_{k=-\infty}^{\infty} g(x+2\pi k)$.

Then h is 2π-periodic and its Fourier coefficients are $\widehat{h}_n = \frac{1}{2\pi}\int_0^{2\pi} h(x)e^{-inx}\,dx$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{2\pi} g(x+2\pi k)e^{-inx}\,dx = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{2\pi k}^{2\pi(k+1)} g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-inx}\,dx = \frac{1}{2\pi}\widehat{g}(n)$$

Theorem: $\{\varphi(x-k) : k \in \mathbf{Z}\}$ is an orthonormal set $\Leftrightarrow \sum_{k=-\infty}^{\infty} |\widehat{\varphi}(\omega+2\pi k)|^2 = 1$

Proof: Let $\varphi \in L^2(\mathbf{R})$ and let $\varphi_k = \varphi(x-k)$.

Let $g(\omega) = |\widehat{\varphi}|^2$ and $h(\omega) = \sum_{k=-\infty}^{\infty} g(\omega+2\pi k) = \sum_{k=-\infty}^{\infty} |\widehat{\varphi}(\omega+2\pi k)|^2$.

Then h is 2π-periodic and by the Poisson summation formula, its Fourier coefficients are $\widehat{h}_n = \frac{1}{2\pi}\widehat{g}(n)$.

Now $\widehat{\varphi}_k = \int_{-\infty}^{\infty} \varphi(x-k)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \varphi(x)e^{-i\omega(x+k)}\,dx = e^{-i\omega k}\widehat{\varphi}$, so by Plancherel's theorem

$$\langle \varphi_m, \varphi_n \rangle = \frac{1}{2\pi}\langle \widehat{\varphi}_m, \widehat{\varphi}_n \rangle = \frac{1}{2\pi}\left\langle e^{-i\omega m}\widehat{\varphi}, e^{-i\omega n}\widehat{\varphi}\right\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(m-n)}\,|\widehat{\varphi}|^2\,d\omega = \frac{1}{2\pi}\widehat{g}(m-n) = \widehat{h}_{m-n}$$

Thus, $\langle \varphi_m, \varphi_n \rangle = \delta_{mn} \Leftrightarrow \widehat{h}_0 = 1$ and $\widehat{h}_n = 0$ for $n \neq 0 \Leftrightarrow h(\omega) \equiv 1$