Path independence and potential

Equivalence of path independence and the existence of a potential: Given a vector field F, sometimes we can guess a function f such that $df = F \cdot ds$ (i.e. $D(f) = F$). Such a function is called a *potential* for F (except in electromagnetic theory, where by convention there is an extra minus ($E = -\nabla V$) arising from electrons being negatively charged). If a potential exists, F is called *conservative* (its energy is conserved) or a *gradient field* ($F = \nabla f$) and the corresponding differential form $F \cdot ds$ is called *exact*. In this case path independence follows from F.T.C. (i). Conversely, F.T.C. (ii) gives a potential (the indefinite integral) if we have path independence.

A necessary condition for a potential: If we have a potential, then $\partial F_i/\partial x_j = \partial F_j/\partial x_i$ for $i \neq j$. If a vector field F satisfies this condition, the corresponding differential form $F \cdot ds$ is called *closed* (cf. Th. 6.3.5, pp. 400, 401). Another way of expressing the necessity of this condition is: exact forms are closed.

Sketch of proof: $\partial F_i/\partial x_j = \partial\left(\partial f/\partial x_i\right)/\partial x_j = \partial\left(\partial f/\partial x_j\right)/\partial x_i = \partial F_j/\partial x_i$ by the equality of mixed partial derivatives (Th. 2.4.3, pp. 138, 140).

A criterion for path independence: In general, path independence is impossible to verify directly (cf. top of p. 397). However, at least in the case when the domain of F is *simply connected*, i.e. all closed curves can be continuously contracted inside the domain to a point (cf. Def. 6.3.4, p. 399), the above necessary condition is actually sufficient for the existence of a potential (cf. Th. 6.3.5, pp. 400, 401). Another way of saying this is: on a simply connected domain closed forms are exact. This is sometimes known as Poincaré's Lemma, although it is due to V. Volterra.

Sketch of proof: To get an idea of how this is proved let us make a simplifying assumption that we are in $\mathbf{R}^2$. F.T.C. (ii) suggests that an integral is a good candidate for a potential and since ultimately we seek path independence, an easy choice of a path may be sufficient. Consider the straight line segment from the origin to an arbitrary point (x, y) parametrized by $s(t) = (x, y)t = (xt, yt), 0 \le t \le 1$. Define our candidate for a potential by

$$f(x,y) = \int F \cdot ds = \int F_1(xt,yt)\,d(xt) + F_2(xt,yt)\,d(yt) = \int_0^1 F_1(xt,yt)x\,dt + F_2(xt,yt)y\,dt$$

and take the partial derivative (under the integral sign) with respect to x (the other partial derivative is similar):

$$\frac{\partial f(x,y)}{\partial x} = \int_0^1 \frac{\partial F_1(xt,yt)}{\partial x}x\,dt + F_1(xt,yt)\,dt + \frac{\partial F_2(xt,yt)}{\partial x}y\,dt.$$

Now use the chain rule (derivatives with respect to the first variable are denoted by D_1 to avoid confusion with $\partial/\partial x$):

$$\frac{\partial f(x,y)}{\partial x} = \int_0^1 D_1(F_1(xt,yt))\,tx\,dt + F_1(xt,yt)\,dt + D_1(F_2(xt,yt))\,ty\,dt.$$

By our assumption $D_1(F_2) = D_2(F_1)$, so

$$\frac{\partial f(x,y)}{\partial x} = \int_0^1 D_1(F_1(xt,yt))\,tx\,dt + F_1(xt,yt)\,dt + D_2(F_1(xt,yt))\,ty\,dt = \int_0^1 \frac{\partial F_1(xt,yt)\,t}{\partial t}\,dt.$$

By F.T.C. (i) we obtain

$$\frac{\partial f(x,y)}{\partial x} = F_1(xt,yt)\,t\Big|_0^1 = F_1(x,y).$$

Finding a potential: A potential can be found by several techniques.

(i) The most obvious technique is guessing a formula for f and checking that $df = F \cdot ds$ (cf. Example 3, p. 398).

(ii) A more algorithmic technique is provided by indefinite integration as in F.T.C. (ii). Integration can be difficult, although sometimes a judicious choice of path may be of assistance (cf. Example 6, p. 400).

(iii) Another common technique is partial integration illustrated in Examples 7 and 8 (p. 402).

Application to integration: Given a differential form $F \cdot ds$, it is easy to check whether it is closed. If so, and if the domain of definition of F is simply connected, $F \cdot ds$ is exact, so there exists a potential. Any integral of $F \cdot ds$ can now be evaluated using F.T.C. (i) — it is simply the difference in potential between the endpoints of the path (cf. Example 3, p. 398). For a closed path the integral would then be 0 (cf. Example 5, p. 400).

Reference: S. J. Colley, *Vector Calculus*, Prentice-Hall, 1999.