Normal convergence

Uniform convergence: A sequence f_n converges uniformly on $K \subseteq \mathbf{C}$ means $\forall \varepsilon > 0\ \exists N\ \forall n \geq N \sup_{z \in K} |f(z) - g(z)| < \varepsilon$.

Weierstrass M-test: If $\forall z \in K\ \ |f_n(z)| \leq M_n$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on K.

Normal convergence: Given a domain $\Omega \subseteq \mathbf{C}$, a sequence of functions $f_n : \Omega \to \mathbf{C}$ converges *locally* uniformly means $\forall z \in \Omega\ \exists \delta > 0$ such that the functions f_n restricted to $B_\delta(z)$ converge uniformly.
Heine-Borel theorem implies that locally uniform convergence is equivalent to convergence that is uniform on compact subsets. Topologists call this *compact* convergence, while complex analysts call it *normal* convergence.

Compact-open topology: Let X and Y be topological spaces and $C(X, Y)$ be the set of all continuous functions $X \to Y$. For each compact $K \subseteq X$ and open $U \subseteq Y$ let $S = \{f\colon f(K) \subseteq U\}$. The topology on $C(X, Y)$ generated by all such S is called the *compact-open* topology. In this topology functions are near when their values are close on compact sets.
The compact-open topology on $C(\Omega, \mathbf{C})$ is exactly the topology of normal convergence (see Theorems XII.7.2 [3], 5.1 [4]).

The space of holomorphic functions: Let $\mathcal{H}(\Omega)$ denote the space of holomorphic functions on Ω. We can construct a metric for the compact-open topology on $\mathcal{H}(\Omega) \subseteq C(\Omega, \mathbf{C})$ by writing Ω as a union of a tower of compact subsets and using a bounded uniform metric on these subsets.

Exhaustion by compact sets: A family of compact sets $\{K_n\}_{n \in \mathbf{Z}_+}$ is an *exhaustion* of Ω means $K_n \subseteq \mathring{K}_{n+1}$, $\Omega = \bigcup_{n \in \mathbf{Z}_+} K_n$, and for all compact $K \subseteq \Omega\ \exists n$ with $K \subseteq K_n$. E.g. let $K_n = \{z \in \Omega\colon |z| \leq 1, d(z, \mathbf{C} \setminus \Omega) \geq 1/n\}$ (see §2.2 [1])

Metric: On each K_n let u_n be the uniform metric, i.e. $u_n(f, g) = \sup_{z \in K_n} |f(z) - g(z)|$. Now let d_n be a bounded uniform metric, e.g. $d_n = \dfrac{u_n}{1 + u_n}$ or $d_n = \inf\{1, u_n\}$. Finally, define a metric on $\mathcal{H}(\Omega)$ by $d = \sum_{n=1}^{\infty} 2^{-n} d_n$.
Normal convergence is equivalent to convergence with respect to d (see Theorem 1.3.2 [5]).

Termwise integration: Let L be a rectifiable curve in Ω. If f_n is a normally convergent sequence in $\mathcal{H}(\Omega)$, then the limit f is continuous (see Theorems 9.2 [6], 4.4 [4]), thus integrable on L. Since L is compact, $f_n \to f$ uniformly on L, so $\int_L f_n(z)\, dz \to \int_L f(z)\, dz$ (see Theorem 9.3 [6]).
Proof: $\left| \int_L f_n(z)\, dz - \int_L f(z)\, dz \right| \leq \int_L |f_n(z) - f(z)|\ |dz| \leq \sup_{z \in L} |f_n(z) - f(z)| \int_L |dz| = u_L(f_n, f)\, |L| \to 0$ as $n \to \infty$.

Weierstrass theorem: $\mathcal{H}(\Omega)$ is a Fréchet space (complete metric space) (see Theorems 9.4 [6], 2.2.1 [1], VII.2.1 [2]).
Proof: If $z_0 \in \Omega$, $\exists \delta > 0\ B_\delta(z_0) \subseteq \Omega$. Let $L \subseteq B_\delta(z_0)$ be a closed rectifiable curve. Suppose $f_n \in \mathcal{H}(\Omega)$ and $f_n \to f$ normally. Integrate termwise and apply Cauchy's theorem to obtain $\int_L f(z)\, dz = 0$. By Morera's theorem f is holomorphic on $B_\delta(z_0)$, so $f \in \mathcal{H}(\Omega)$.

Termwise differentiation: If $f_n \to f$ normally, then $f_n^{(k)} \to f^{(k)}$.
Proof: Use termwise integration and Cauchy's Integral Formula $\displaystyle\int_L \frac{f(z)\, dz}{(z - z_0)^{k+1}} = \frac{2\pi i}{k!} f^{(k)}(z_0)$.

Taylor series: If $\sum_{k=0}^{\infty} a_k (z - z_0)^k \to f(z)$ on $B_\delta(z_0)$, then by the Weierstrass M-test the convergence is normal, so f is holomorphic on $B_\delta(z_0)$. Termwise differentiation shows that $a_k = f^{(k)}(z_0)/k!$. Conversely suppose f is holomorphic at z_0 and let L be a circle centered at z_0 such that f is holomorphic inside L. Then for z inside L, $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_L \frac{f(w)\, dw}{w - z} =$

$$\frac{1}{2\pi i} \int_L \frac{1}{1 - \left(\dfrac{z - z_0}{w - z_0}\right)} \frac{f(w)\, dw}{(w - z_0)} = \frac{1}{2\pi i} \int_L \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^k \frac{f(w)\, dw}{(w - z_0)} = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_L \frac{f(w)\, dw}{(w - z_0)^{k+1}}\right) (z - z_0)^k = \sum_{k=0}^{\infty} a_k\, (z - z_0)^k.$$

Thus, $f \in \mathcal{H}(\Omega) \Leftrightarrow f$ can be locally expanded in a power series (analytic) (see §48 [6], 2.2.2 [1], §IV.2 [2]).

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