Differentiation in $\mathbf{R}^3$

Differentials: Given $f(r)$ on $\mathbf{R}^3$ and a point r_0 let $\Delta r = r - r_0$. The differential df is the linear function of Δr whose graph is tangent to the graph of $w = f(r)$ at r_0. If we choose the coordinate projections as the basis for the vector space of linear maps of Δr, then we can expand df in this basis. We denote the coordinate projections by dx, dy, dz (e.g. $dy(\Delta r) = \Delta y$). The coefficients are called the *partial derivatives* of f at r_0 and we get $df = f_x\,dx + f_y\,dy + f_z\,dz = D(f)\,dr$

Forms: A linear function of Δr that also depends on r_0 (from now on we will drop the subscript) is called a 1-form. Thus, df is a 1-form. Higher degree forms are multilinear alternating functions. An n-form is a function of n-variables, linear in each variable, such that interchanging variables produces a minus sign (in general, a permutation of the variables gives its parity). For example, $\Delta r_1, \Delta r_2 \mapsto \Delta y_1\,\Delta z_2 - \Delta y_2\,\Delta z_1 = \det\begin{pmatrix}\Delta y_1 & \Delta y_2\\ \Delta z_1 & \Delta z_2\end{pmatrix}$ is a 2-form denoted by $dy\,dz$.

The vector space of all n-forms is denoted Λ^n. The table below shows cartesian expansions of n-forms on $\mathbf{R}^3$:

degree	name	cartesian coordinate form	dim Λ^n
0-form	function	$f = f(x,y,z)$	1
1-form	work form	$\omega = A(x,y,z)\,dx + B(x,y,z)\,dy + C(x,y,z)\,dz$	3
2-form	flux form	$\varphi = P(x,y,z)\,dy\,dz + Q(x,y,z)\,dz\,dx + R(x,y,z)\,dx\,dy$	3
3-form	density form	$\rho = f(x,y,z)\,dx\,dy\,dz$	1

Products: In the above example $dy\,dz = -dz\,dy$. Also $dx\,dx = 0$. These are general principles which we can apply to multiplication of forms. Famous vector products are special cases of this multiplication.

product of forms	vector interpretation
$u\,v = (u_x dx + u_y dy + u_z dz)\,(v_x dy\,dz + v_y dz\,dx + v_z dx\,dy) = (u_x v_x + u_y v_y + u_z v_z) dx\,dy\,dz$	dot product
$u\,v = (u_x\,dx + u_y\,dy + u_z\,dz)\,(v_x\,dx + v_y\,dy + v_z\,dz) = (u_y v_z - u_z v_y)\,dy\,dz + (u_z v_x - u_x v_z)\,dz\,dx$ $+ (u_x v_y - u_y v_x)\,dx\,dy = \det\begin{pmatrix}u_y & u_z\\ v_y & v_z\end{pmatrix} dy\,dz + \det\begin{pmatrix}u_z & u_x\\ v_z & v_x\end{pmatrix} dz\,dx + \det\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix} dx\,dy$	cross product
$u\,v\,w = \det\begin{pmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{pmatrix} dx\,dy\,dz$	triple product

Differentials of n-forms: We extend the definition of d from 0-forms to n-forms by imposing the rules of differentiation

linearity		product rule	
$dc = 0$	$d(\omega + \eta) = d\omega + d\eta$	$d(\omega\,\eta) = d\omega\,\eta + (-1)^{\deg\omega}\omega\,d\eta$	$d(d(\omega)) = 0$

The tables below show d of various forms; the correspoding vector differential operators, where we use the del operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and the usual vector products, except that the partials are *applied*; and some of the vector versions of the rules.

differential	vector interpretation
$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$	$\operatorname{grad} f = D(f) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
$d\omega = d(Adx + Bdy + Cdz) = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy\,dz$ $+ \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz\,dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx\,dy$	$\operatorname{curl}\Psi = \operatorname{rot}\Psi = \nabla \times \Psi$ $= \left(\frac{\partial \Psi_z}{\partial y} - \frac{\partial \Psi_y}{\partial z}, \frac{\partial \Psi_x}{\partial z} - \frac{\partial \Psi_z}{\partial x}, \frac{\partial \Psi_y}{\partial x} - \frac{\partial \Psi_x}{\partial y}\right)$
$d\varphi = d(Pdy\,dz + Qdz\,dx + Rdx\,dy)$ $= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$	$\operatorname{div}\Phi = \nabla \cdot \Phi = \frac{\partial \Phi_x}{\partial x} + \frac{\partial \Phi_y}{\partial y} + \frac{\partial \Phi_z}{\partial z}$

$\nabla(f+g) = \nabla f + \nabla g$
$\nabla \times (\Psi_1 + \Psi_2) = \nabla \times \Psi_1 + \nabla \times \Psi_2$
$\nabla \cdot (\Phi_1 + \Phi_2) = \nabla \cdot \Phi_1 + \nabla \cdot \Phi_2$

$\nabla(fg) = (\nabla f)g + f(\nabla g)$
$\nabla \times (f\Psi) = (\nabla f) \times \Psi + f(\nabla \times \Psi)$
$\nabla \cdot (\Psi_1 \times \Psi_2) = (\nabla \times \Psi_1) \cdot \Psi_2 - \Psi_1 \cdot (\nabla \times \Psi_2)$
$\nabla \cdot (f\Phi) = (\nabla f) \cdot \Phi + f(\nabla \cdot \Phi)$

$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \Phi) = 0$

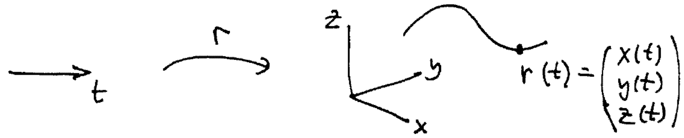
Poincaré's lemma: [1] For *contractible* [2] domains we have a converse to $d(d(\omega)) = 0$, namely if $d\varphi = 0$, then there is ω such that $d\omega = \varphi$. Famous special cases of this say that an irrotational (conservative) vector field has a potential and a divergence-free vector field has a vector potential

[1] Due to Vito Volterra (1860–1940).

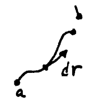
[2] A space is *contractible* means the identity map on this space is homotopic to the constant map. Roughly speaking this means that the space is continuously deformable to a point. Since various "holes" in space obstruct such a deformation, a contractible space can be thought of as lacking holes. A *star-shaped* domain is contractible.

Integration in $\mathbf{R}^3$

Curves: A curve is parametrized by a continuous function of one parameter $r\colon [a,b] \to \mathbf{R}^n$



$dr = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = D(r)\,dt = r'dt$ is tangent to the curve at r:

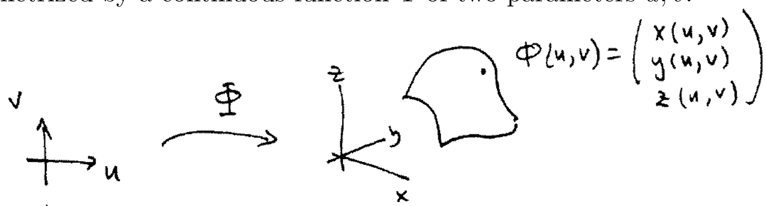


Unit length: $|dr| = \sqrt{dx^2 + dy^2 + dz^2}$

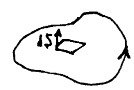
Integration of a vector field (a work form) along a curve: $\int F \cdot dr = \int F_x\,dx + F_y\,dy + F_z\,dz$

Integration of a scalar field along a curve: $\int f\,|dr|$ Special case: arc length $\int |dr|$

Surfaces: A surface is parametrized by a continuous function Φ of two parameters u, v.



$dS = \begin{pmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{pmatrix} = \begin{pmatrix} \det(D(y,z)) \\ \det(D(z,x)) \\ \det(D(x,y)) \end{pmatrix} du\,dv = \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) du\,dv$ is $\perp$ to the surface:

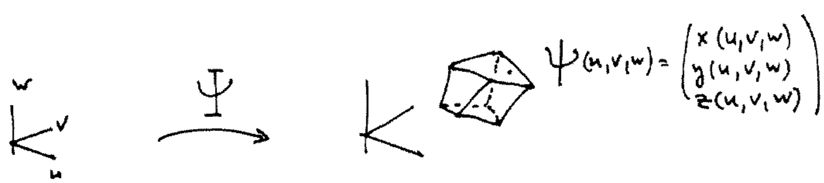


Unit area: $|dS|$

Integration of a vector field (a flux form) through a surface: $\int F \cdot dS = \int F_x\,dy\,dz + F_y\,dz\,dx + F_z\,dx\,dy$

Integration of a scalar field on a surface: $\int f\,|dS|$ Special case: surface area $\int |dS|$

Solids: A solid is parametrized by a continuous function Ψ of three parameters u, v, w.



$dV = dx\,dy\,dz = \det(D(\Psi))\,du\,dv\,dw = \left(\frac{\partial \Psi}{\partial u}\frac{\partial \Psi}{\partial v}\frac{\partial \Psi}{\partial w}\right) du\,dv\,dw$

Unit volume: $|dV|$

Integration of a scalar field (a density form) over a volume: $\int f dV = \int f\,dx\,dy\,dz$

Fundamental Theorem of Calculus:

If ω is a smooth n-form on an n-dimensional domain Ω with smooth boundary $\partial\Omega$, then $\boxed{\int_\Omega d\omega = \int_{\partial\Omega} \omega}$

Famous special cases:

Barrow's rule [3] (incl. F.T.C. for $\mathbf{R}$): $\int \nabla f \cdot dr = f(b) - f(a)$

Stokes' theorem [4] (incl. Green's theorem in the plane): $\iint_D (\nabla \times F) \cdot dS = \int_{\partial D} F \cdot dr$

Gauss-Ostrogradski divergence theorem: $\iiint_B (\nabla \cdot F)\,dV = \iint_{\partial B} F \cdot dS$

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[3] Isaac Barrow (1630–1677) was the first to recognize that integration and differentiation were inverse operations. In 1669 Barrow resigned as Lucasian professor of mathematics at Cambridge in favour of his pupil Newton.

[4] This theorem first appeared in a letter of July 2, 1850 from William Thomson (1824–1907) (Baron Kelvin of Largs, 1892) to George Gabriel Stokes (1819–1903), Lucasian Professor of Mathematics at Cambridge (1849), who included it in his next exam.