From Laurent to Fourier

Suppose $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ is a Laurent series convergent on an annulus $\{z : 0 < |z| < R\}$.

If $R > 1$, the series converges on S^1, so a substitution $z = e^{i\theta}$ gives Fourier series $u(\theta) := f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$.

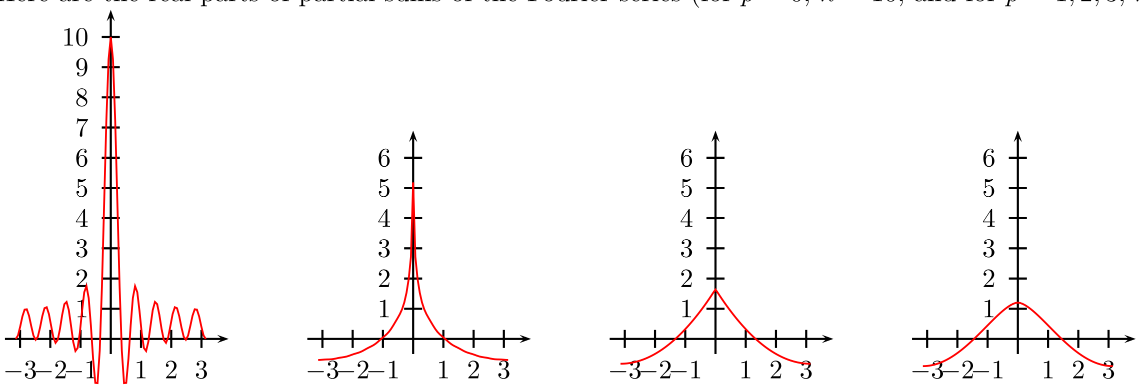
Real version: Using Euler's formula, we may rewrite $c_{-k}e^{-ik\theta} + c_k e^{ik\theta}$ as a linear combination of $\cos(k\theta)$ and $\sin(k\theta)$. We may write $u(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)$. For real valued u, complexophobes use this form to avoid complex numbers altogether. Exercise: what is the relationship between a_k, b_k and c_k?

Question: What happens if $R = 1$?

Example: $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$. We have a singularity at 1 and the radius of convergence is 1.

Likewise, the radius of convergence will be 1 for $\sum_{k=0}^{\infty} \frac{1}{k^p} z^k$ for $p = 1, 2, 3$.

Here are the real parts of partial sums of the Fourier series (for $p = 0$, $n = 10$, and for $p = 1, 2, 3$, $n = 100$).

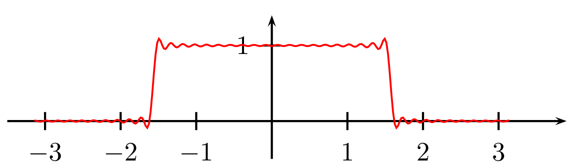


For $k = 0$, if we take larger partial sums, the peak at 0 increases, the frequency increases, but the maxima near $\pm\pi$ stay at around 1. For $p = 1, 2, 3$ we see something resembling convergence. The resulting function is seems smoother for larger p. [1]

Questions:

- Which periodic functions are representable by Fourier series, and in what sense?
- Given $u(\theta)$ how do we find c_k? Answer: $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\theta)e^{-ik\theta}\,d\theta$. Exercise: find formulas for a_k and b_k.

Example: Let $u(\theta) = 1$ for $-\pi/2 \le \theta \le \pi/2$ and 0 otherwise. Below is a partial sum ($n = 40$) of its Fourier series. By taking larger sums one can see how the convergence is not uniform near the points of discontinuity — we get overshoots, which get thinner, yet whose amplitude does not diminish as n increases. [2] For all n, the partial sum goes through the average of left and right limits.



A brief history of Fourier series:

- 1700 Sauveur — experiments with harmonics
- 1740 Daniel Bernoulli (St. Petersburg) — superposition of harmonics
- 1747 Controversy between Euler and d'Alembert
- 1807 Fourier — formula for coefficients
- 1829 Dirichlet — first convergence proof
- 1965 Carleson — almost everywhere pointwise convergence for square summable functions

[1] This is a general, and extremely useful, feature of Fourier series — the faster the coefficients go to 0, the smoother the result.
[2] This is known as the Gibbs phenomenon, discovered about a century before Gibbs.