Attracting fixed points of an iteration:

Let Ω be a domain and $z^* \in \Omega$. Suppose $\varphi \in \mathcal{H}(\Omega)$ such that $\varphi(z^*) = z^*$ and $|\varphi'(z^*)| < 1$. We will show that z^* is an attractor of φ.

By definition of differentiability we have a linear approximation

$$\varphi(z) - \varphi(z^*) = \varphi'(z^*)(z - z^*) + \varepsilon(z), \quad \text{where } \frac{\varepsilon(z)}{z - z^*} \to 0 \text{ as } z \to z^* \tag{1}$$

Choose k with $|\varphi'(z^*)| < k < 1$ and let $\delta = k - |\varphi'(z^*)|$. Then $\delta > 0$, so choose a disc D around z^* such that for all $z \in D \setminus \{z^*\}$ we have $|\varepsilon(z)/(z - z^*)| < \delta$. Thus, for all $z \in D$ we have $|\varepsilon(z)| \le \delta\,|z - z^*|$ and equation (1) gives

$$|\varphi(z) - \varphi(z^*)| \le |\varphi'(z^*)|\,|z - z^*| + |\varepsilon(z)| \le \left(|\varphi'(z^*)| + \delta\right)|z - z^*| = k\,|z - z^*|$$

If $z_0 \in D$, then the orbit $z_n = \varphi(z_{n-1})$ converges to z^*, because

$$|\varphi(z_n) - z^*| = |\varphi(z_n) - \varphi(z^*)| \le k\,|z_n - z^*| = k\,|\varphi(z_{n-1}) - z^*| \le \ldots \le k^n\,|z_0 - z^*| \to 0 \text{ as } n \to \infty$$

Multiplicity:

Suppose $f(z^*) = 0$ and f has a Taylor series at z^* with coefficients $c_n = f^{(n)}(z^*)/n!$. Multiplicity of z^* is defined to be $m = \min\{n : c_n \ne 0\}$. Every term in the series is divisible by $(z - z^*)^m$, so $g(z) = f(z)/(z - z^*)^m$ is analytic at z^* and $g(z^*) \ne 0$.

Newton's method:

History: In 1669 Newton introduced successive linearization for finding roots of cubic equations. The usual formula with the derivative is due to Joseph Raphson (1690). The method was first applied to complex functions by Cayley in 1879.

Let $\varphi(z) = z - f(z)/f'(z)$ if $f'(z) \ne 0$, and $\varphi(z) = z$ if $f'(z) = 0$. Suppose $f(z^*) = 0$. Since the zeros of f' are isolated we can find a punctured disc around z^* where f' is nonzero. Thus, we may assume that the second clause in the definition of φ at worst applies only at z^*. We have $\varphi(z^*) = z^*$ and we will show that $|\varphi'(z^*)| < 1$.

If z^* is a simple root of f, then $f'(z^*) \ne 0$ and $f(z^*) = 0$, so $\varphi'(z^*) = f(z^*)f''(z^*)/f'(z^*)^2 = 0$. In this case we may choose arbitrarily small k and obtain superlinear convergence (we will see later that it is at least quadratic).

If z^* is a root of f with multiplicity $m > 1$, then $f'(z^*) = 0$, so computing $\varphi'(z^*)$ is trickier. Write $f(z) = (z - z^*)^m g(z)$, where $g(z^*) \ne 0$. Differentiate $\log f(z) = m \log(z - z^*) + \log g(z)$ to get

$$\frac{f'(z)}{f(z)} = \frac{m}{(z - z^*)} + \frac{g'(z)}{g(z)}, \quad \text{so } \frac{(z - z^*)f'(z)}{f(z)} = m + \frac{(z - z^*)g'(z)}{g(z)} \to m \text{ as } z \to z^*$$

Therefore,

$$\varphi'(z^*) = \lim_{z \to z^*} \frac{\varphi(z) - \varphi(z^*)}{z - z^*} = \lim_{z \to z^*} \frac{[z - f(z)/f'(z)] - z^*}{z - z^*} = 1 - \lim_{z \to z^*} \frac{f(z)}{(z - z^*)f'(z)} = 1 - \frac{1}{m} < 1$$

Superconvergence to simple roots:

If $\varphi'(z^*) = 0$, then z^* is a multiple root of $z^* - \varphi(z)$, so write $z^* - \varphi(z) = (z - z^*)^m h(z)$, where $h(z^*) \ne 0$ and $m > 1$. By a slight shrinking of D we can obtain continuity of h on $\overline{D}$, so we may assume that $|h| \le M$ on D.

For $z_0 \in D$ look at its orbit $z_n = \varphi(z_{n-1})$, and consider the absolute error $\varepsilon_n = z^* - z_n$. Then

$$|\varepsilon_{n+1}| = |z^* - z_{n+1}| = |z^* - \varphi(z_n)| = |z_n - z^*|^m\,|h(z_n)| = |h(z_n)|\,|\varepsilon_n|^m \le M\,|\varepsilon_n|^m$$

Reference: Alan F. Beardon, *Iteration of Rational Functions*, Springer-Verlag, 1991