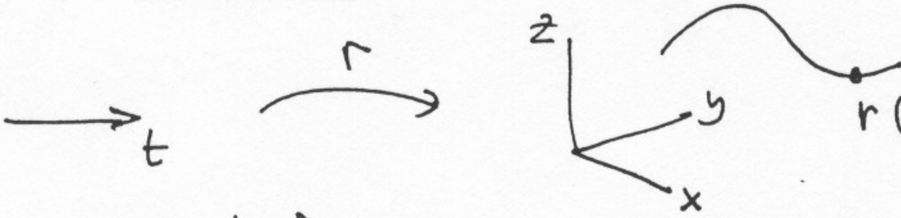
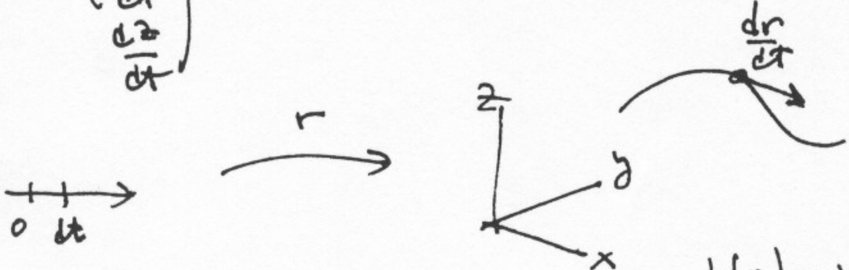
CURVES



$$r(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$$dr = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$ Evaluate at r:

$$dr = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} dt$$



$|dr| =$ unit length

Integration along a curve

$$\int F \cdot dr = \int F_x\,dx + F_y\,dy + F_y\,dz = \int F \cdot \frac{dr}{dt}\,dt$$ (cf. p. 356, 357

$$\int f \cdot |dr| = \int f \left| \frac{dr}{dt} \right| dt$$ (cf. p. 370)

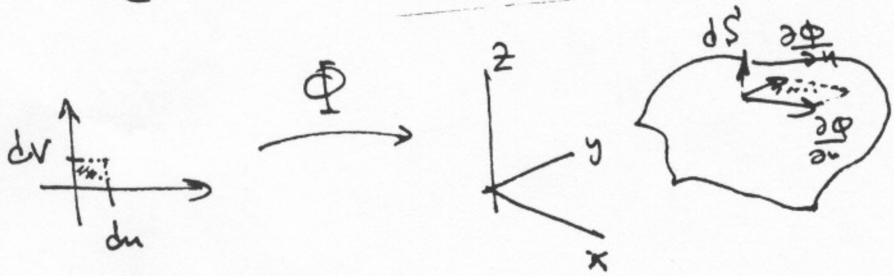
SURFACES

$$\Phi(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$dS = \begin{pmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{pmatrix}$$ Evaluate at Φ:

$$dS = \begin{bmatrix} \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right) \\ \left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right)\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) \\ \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) \end{bmatrix} = \begin{pmatrix} \left(\frac{\partial y}{\partial u}\cdot\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial u}\right) \\ \left(\frac{\partial z}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\cdot\frac{\partial x}{\partial u}\right) \\ \left(\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}\right) \end{pmatrix} du\,dv$$

$$= \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) du\,dv$$ (cf. p. 387)



(cf. p. 387, 398)

Tangent vectors to the surface: $\frac{\partial \Phi}{\partial u}$, $\frac{\partial \Phi}{\partial v}$

$dS \perp$ to the surface

$|dS| = \left|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right|$ = unit area. (cf. p. 387)

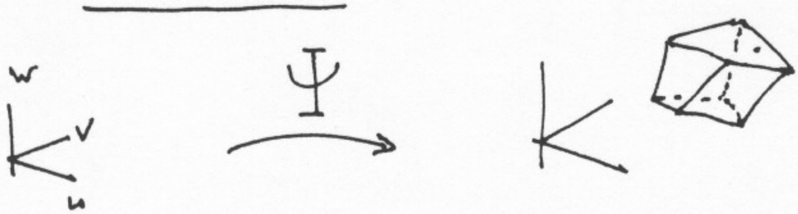
Integration over a surface

$$\int F \cdot dS = \int F_x\,dy\,dz + F_y\,dz\,dx + F_z\,dx\,dy = \int F \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) du\,dv$$

(cf. p. 398)

$$\int f\,|dS| = \int f\,\left|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right| du\,dv$$ (cf. p. 393)

VOLUMES



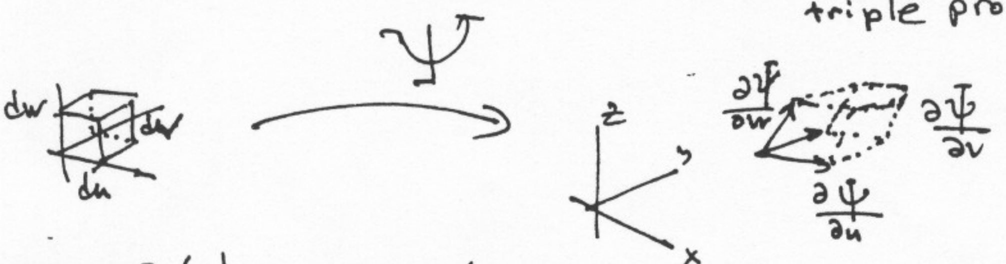
$$\Psi(u,v,w) = \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{pmatrix}$$

$dV = dx\,dy\,dz$ Evaluate at Ψ:

$$dV = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right)\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw\right)\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw\right)$$

$$= \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial w}\,du\,dv\,dw + \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\,\frac{\partial z}{\partial w}\,dv\,du\,dw + \ldots =$$

$$= \det(\Psi')\,du\,dv\,dw = \underbrace{\left(\frac{\partial \Psi}{\partial u}\;\frac{\partial \Psi}{\partial v}\;\frac{\partial \Psi}{\partial w}\right)}_{\text{triple product}}du\,dv\,dw$$



$|\det(\Psi')|$ = unit volume

Integration

$$\int f\,dV = \int f\,dx\,dy\,dz = \int f\,\det(\Psi')\,du\,dv\,dw$$

(cf. p. 336)

$$\{f:\mathbb{R}^3\to\mathbb{R}\} \xrightarrow{d} \{P\,dx+Q\,dy+R\,dz\} \xrightarrow{d} \{A\,dy\,dz+B\,dz\,dx+C\,dx\,dy\} \xrightarrow{d} \{E\,dx\,dy\,dz\}$$

- $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = f'\cdot dr = \nabla f\cdot dr$ (gradient)

- $d(P\,dx+Q\,dy+R\,dz) = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)dx +$

$$+\left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right)dy + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right)dz =$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy =$$

$$= \left[\nabla\times\begin{pmatrix}P\\Q\\R\end{pmatrix}\right]\cdot d\vec{S} \qquad \text{(curl)}$$

- $d(A\,dy\,dz + B\,dz\,dx + C\,dx\,dy) = \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz\right)dy\,dz +$

$$+\left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy + \frac{\partial B}{\partial z}dz\right)dz\,dx + \left(\frac{\partial C}{\partial x}dx + \frac{\partial C}{\partial y}dy + \frac{\partial C}{\partial z}dz\right)dx\,dy =$$

$$= \frac{\partial A}{\partial x}dx\,dy\,dz + \frac{\partial B}{\partial y}dy\,dz\,dx + \frac{\partial C}{\partial z}dz\,dx\,dy = \left[\nabla\cdot\begin{pmatrix}A\\B\\C\end{pmatrix}\right]dx\,dy\,dz$$

(divergence)

Poincaré lemma

- $d(d(\omega)) = 0$, i.e. $\underbrace{\gamma = d\omega}_{\gamma\ \text{exact}} \Rightarrow \underbrace{d\gamma = 0}_{\gamma\ \text{closed}}$

  E.g. curl(grad) = 0 (cf. p. 256)

  div(curl) = 0 (cf. p. 258)

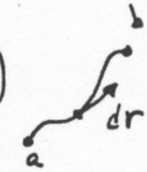
- If the region is star shaped 

  $\underbrace{d\gamma = 0}_{\gamma\ \text{closed}} \Rightarrow \exists\,\omega \quad \underbrace{\gamma = d\omega}_{\gamma\ \text{exact}}$

Fundamental theorem of calculus:
(a.k.a. Stokes theorem)

$$\boxed{\int_{\partial D} \omega = \int_{D} d\omega}$$

<u>Barrow's rule:</u> $\int \nabla f \cdot dr = f(b) - f(a)$ (cf. p. 358)



<u>Stokes theorem:</u> $\int_{\partial D} F \cdot dr = \iint_{D} (\nabla \times F) \cdot d\mathbf{S}$ (cf. p. 424)

<u>Gauss divergence theorem:</u> $\iint_{\partial D} F \cdot d\mathbf{S} = \iiint_{D} (\nabla \cdot F)\, dV$ (cf. p. 446)