Fourier Transform

Fourier series: Given a 2π-periodic complex-valued function f (think of it as a function on an interval $\mathbf{T}$ of length 2π), its Fourier (series) transform is the sequence of its Fourier coefficients: $\mathscr{F}(f) = \widehat{f} = \{c_k \colon k \in \mathbf{Z}\}$, which can be thought of as a complex-valued function of the discrete frequency variable k.

Conversely, the inverse Fourier transform of a sequence c_k is the function given by the Fourier series: $\mathscr{F}^{-1}(\{c_k\}) = f$.

Schematically this can be represented as follows:

$$\begin{array}{rcl} f : \mathbf{T} & \to & \mathbf{C} \\ t & \longmapsto & f(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ikt} \end{array} \quad \overset{\mathscr{F}}{\rightsquigarrow} \quad \begin{array}{rcl} \widehat{f} : \mathbf{Z} & \to & \mathbf{C} \\ k & \longmapsto & c_k = \dfrac{1}{2\pi}\, f(t) \cdot e^{ikt} = \dfrac{1}{2\pi} \displaystyle\int_{\mathbf{T}} f(t)\, e^{-ikt}\, dt \end{array}$$

If f is real-valued, it is possible to avoid complex numbers altogether:

$$f(t) = a_0 + \sum_{k=0}^{\infty} \left[a_k \cos(kt) + b_k \sin(kt)\right] \qquad a_0 = \frac{1}{2\pi}\, f(t) \cdot 1 = \frac{1}{2\pi} \int_{\mathbf{T}} f(t)\, dt$$

$$a_k = \frac{1}{\pi}\, f(t) \cdot \cos(kt) = \frac{1}{\pi} \int_{\mathbf{T}} f(t)\, \cos(kt)\, dt$$

$$b_k = \frac{1}{\pi}\, f(t) \cdot \sin(kt) = \frac{1}{\pi} \int_{\mathbf{T}} f(t)\, \sin(kt)\, dt$$

By scaling the independent variable t, similar formulas may be derived for functions of other periods (see next page).

If, in addition, f is even then all $b_k = 0$ and a_k can be computed by doubling the integrals over half the period.

Similarly for odd functions all $a_k = 0$ and b_k can be computed by doubl ing the integrals over half the period.

Fourier transform: If f is not periodic, but satisfies some decay conditions, we can take its Fourier transform $\mathscr{F}(f) = \widehat{f} = C(w)$, which can be thought of as a complex-valued function of a real frequency variable w:

$$\begin{array}{rcl} f : \mathbf{R} & \to & \mathbf{C} \\ t & \longmapsto & f(t) = \displaystyle\int_{-\infty}^{\infty} C(w)\, e^{iwt}\, dw \end{array} \quad \overset{\mathscr{F}}{\rightsquigarrow} \quad \begin{array}{rcl} \widehat{f} : \mathbf{R} & \to & \mathbf{C} \\ w & \longmapsto & C(w) = \dfrac{1}{2\pi}\, f(t) \cdot e^{iwt} = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \end{array}$$

Notice the constant factor $\frac{1}{2\pi}$. For the sake of symmetry we may spread the wealth and define

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w) e^{iwt}\, dw \qquad \widehat{f}(w) = \frac{1}{\sqrt{2\pi}} f(t) \cdot e^{iwt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

Again, if f is real-valued, it is possible to avoid complex numbers:

$$f(t) = \int_0^{\infty} \left[A(w)\, \cos(wt) + B(w)\, \sin(wt)\right] dw \qquad A(w) = \frac{1}{\pi}\, f(t) \cdot \cos(wt) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\, \cos(wt)\, dt$$

$$B(w) = \frac{1}{\pi}\, f(t) \cdot \sin(wt) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\, \sin(wt)\, dt$$

If, in addition, f is even, then $B(w) = 0$ and $A(w) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} f(t)\, \cos(wt)\, dt$.

For the sake of symmetry, we may define the Fourier cosine transform $\mathscr{F}_c$ by

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \widehat{f}_c(w)\, \cos(wt)\, dw \qquad \widehat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\, \cos(wt)\, dt$$

Similarly, if f is odd, then $A(w) = 0$ and $B(w) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} f(t)\, \sin(wt)\, dt$, so define the sine transform $\mathscr{F}_s$:

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \widehat{f}_s(w)\, \sin(wt)\, dw \qquad \widehat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\, \sin(wt)\, dt$$

Fourier series for functions of arbitrary period

Suppose f is a complex-valued p-periodic function and $\mathbf{T}$ is an interval of length p. Then

$$f(t)=\sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{2\pi i k t}{p}\right) \qquad c_k=\frac{1}{p}\int_{\mathbf{T}} f(t)\exp\left(-\frac{2\pi i k t}{p}\right)dt$$

If f is real-valued we have the real version:

$$f(t)=a_0+\sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2k\pi t}{p}\right)+b_k\sin\left(\frac{2k\pi t}{p}\right)\right] \qquad a_0=\frac{1}{p}\int_{\mathbf{T}} f(t)\,dt$$

$$a_k=\frac{2}{p}\int_{\mathbf{T}} f(t)\cos\left(\frac{2k\pi t}{p}\right)dt$$

$$b_k=\frac{2}{p}\int_{\mathbf{T}} f(t)\sin\left(\frac{2k\pi t}{p}\right)dt$$

Properties of the Fourier Transform

Most of the properties of Fourier transforms are quite general and apply in both the series and integral cases, as well as for the sine and cosine transforms (unless otherwise specified). In fact, many properties are shared by other integral and series transforms as well. In what follows both $\mathscr{F}(f)$ and $\widehat{f}$ denote either the usual Fourier series transform or the symmetric Fourier integral transform of f. Non-symmetric Fourier integral transforms have similar properties, sometimes differing by a constant factor.

Linearity: Both the Fourier transform and its inverse are linear: $\mathscr{F}(af+bg)=a\mathscr{F}(f)+b\mathscr{F}(g)$

Isometry: The Fourier transform and its inverse "preserve" the inner product (up to a constant factor). Thus, they preserve "lengths" and "angles" (in particular, they preserve orthogonality). Preservation of lengths is known as Parseval's formula in the series case and Plancherel's formula in the integral case.

$$\widehat{f}\cdot\widehat{g}=f\cdot g \quad\rightsquigarrow\quad \left\|\widehat{f}\right\|^2=\|f\|^2 \quad\rightsquigarrow\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^2\,dt=\sum_{k=-\infty}^{\infty}|c_k|^2 \qquad \int_{-\infty}^{\infty}|f(t)|^2\,dt=\int_{-\infty}^{\infty}\left|\widehat{f}(w)\right|^2 dw$$

In the real case we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2\,dt=2a_0^2+\sum_{k=0}^{\infty}\left[{a_k}^2+{b_k}^2\right] \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)^2\,dt=\int_0^{\infty}\left[A(w)^2+B(w)^2\right]dw$$

Convolution: Given two functions f and g, or two sequences $A=\{a_k\}$ and $B=\{b_k\}$, their convolutions $f*g$ and $A*B$ are defined by

$$(f*g)(\tau)=\int_{-\infty}^{\infty} f(t)\,g(\tau-t)\,dt \qquad (A*B)_n=\sum_{k=-\infty}^{\infty} a_k b_{n-k}$$

It can be shown directly that convolution is a commutative, associative, and distributive (with respect to $+$) binary operation. The Fourier transform and its inverse convert ordinary multiplication into convolution and vice versa:

$$\mathscr{F}(f*g)=\sqrt{2\pi}\,\mathscr{F}(f)\,\mathscr{F}(g)$$

Shift and delay: Let $f_\omega(t)=e^{i\omega t}f(t)$. Then $\widehat{f_\omega}(w)=\widehat{f}(w-\omega)$. Let $f_\tau(t)=f(t-\tau)$. Then $\widehat{f_\tau}(w)=\widehat{f}(w)\,e^{-iw\tau}$.

Conjugation: $\widehat{\overline{f}}(w)=\overline{\widehat{f}(-w)}$

Differentiation: Integration by parts gives $\mathscr{F}(f')=iw\mathscr{F}(f)$

For the real sine and cosine transforms we have slightly more complicated formulas:

$$\mathscr{F}_c(f')=w\mathscr{F}_s(f)-\sqrt{\frac{2}{\pi}}\,f(0) \qquad \mathscr{F}_s(f')=-w\mathscr{F}_c(f)$$