Fundamental Theorem of Calculus for curve integrals

The Fundamental Theorem of Calculus (F.T.C.) is a general statement about the relationship between integration and differentiation. Roughly speaking they are compositional inverses. For functions of a single variable we have

$$\text{(i)} \int_a^x df = \int_a^x f'(t)\,dt = f(x) - f(a), \qquad \text{(ii)} \frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

Sketch of proof: (i) Let $a = a_0 < a_1 < ... < a_n = x$ be a partition of $[a, x]$ and choose t_i is each subinterval $(a_{i-1} \leq t_i \leq a_i)$. Then we approximate $f'(t_i)$ by $\Delta f/\Delta t = (f(a_i) - f(a_{i-1}))/(a_i - a_{i-1})$ and cancel most terms in the Riemann sum

$$\sum_{i=1}^{n} f'(t_i)(a_i - a_{i-1}) \approx \sum_{i=1}^{n} f(a_i) - f(a_{i-1} = f(x) - f(a_{n-1}) + f(a_{n-1}) - f(a_{n-2}) + ...f(a_1) - f(a) = f(x) - f(a).$$

(ii) Let h be small. Then the numerator of the difference quotient $\int_a^{x+h} g(t)\,dt - \int_a^x g(t)\,dt = \int_x^{x+h} g(t)\,dt \approx g(x)h.$

F.T.C. for curves: F.T.C. holds almost verbatim for functions of several variable and integration along curves:

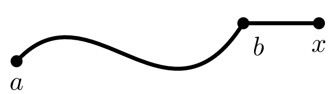
$$\text{(i)} \int_a^x df = \int_a^x D(f)\cdot ds = f(x) - f(a), \qquad \text{(ii)}\ D\left(\int_a^x F\cdot ds\right) = F(x).$$

The main exception is that part (ii) makes sense only for vector fields whose integration is path independent (cf. Def. 6.3.1, p. 395), because otherwise the integral is not solely a function of the vector variable x. Note that path independence follows immediately in part (i) (cf. Th. 6.3.3, pp. 398, 404).

Sketch of proof: (i) Let $s\colon [\alpha, \beta] \to \mathbf{R}^n$ be a smooth parametrized curve with $s(\alpha) = a$ and $s(\beta) = x$. Then $f(s(t))$ is a function of a single variable t, so by the usual F.T.C. (i)

$$\int df = \int_\alpha^\beta d(f(s(t)) = f(s(\beta)) - f(s(\alpha)) = f(x) - f(a).$$

(ii) It is enough to show that the partial derivatives of the integral are precisely the components of F (if F is continuous then so are the partial derivatives of the integral, so its differentiability follows). Let us take partial derivative with respect to x_1 (the others are similar). If the integral is path independent, then we may choose a convenient path, whose last bit is a straight line segment along the x_1 axis, parametrized by $s(t) = (t, b_2, ...b_n), b_1 \leq t \leq x_1$.



Since $ds = (dt, 0, ...0)$, $F \cdot ds = F_1\,dt$. Just as the path is in two parts, so is the integral. The first part is independent of x_1, so differentiates to 0. Applying the usual F.T.C. (ii) to the second part we obtain

$$D_1\left(\int_a^x F\cdot ds\right) = D_1\left(\int_a^b F\cdot ds + \int_b^x F\cdot ds\right) = D_1\left(\int_{b_1}^{x_1} F_1(s(t))\,dt\right) = F_1(s(x_1)) = F_1(x).$$

Reference: S. J. Colley, *Vector Calculus*, Prentice-Hall, 1999.