### Dmitry Gokhman

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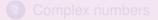
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Outline







Fundamental Theorem of Algebra

Dmitry Gokhman Fundamental Theorem of Algebra

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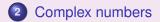
3 Fundamental Theorem of Algebra

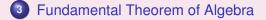
Dmitry Gokhman Fundamental Theorem of Algebra

Outline









Dmitry Gokhman Fundamental Theorem of Algebra

No polynomial of the type  $X^4 + a^4$  (with  $a \neq 0$ ) can be factored. — Gottfried Leibniz (1702) (he was wrong)

 $\bullet$  Intersections of curves  $\longrightarrow$  roots of polynomials

• Quadratics: 
$$z^2 - 2mz + c = 0$$

$$z^2 - 2mz + m^2 = m^2 - c$$
  $(z - m)^2 = m^2 - c$ 

 $z = m \pm \sqrt{m^2 - c}$ 

• Cubics:  $z^3 + 3az^2 + bz + c = 0$ 

(Scipione del Ferro, Niccolò Fontana (Tartaglia), XVI)

To eliminate  $z^2$ , shift the inflection point z = -a to the origin:

 $(z-a)^3 + 3a(z-a)^2 + b(z-a) + c = 0$ 

 $z^{3}-3z^{2}a+3za^{2}-a^{3}+3a(z^{2}-2za+a^{2})+b(z-a)+c=0$ 

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• Depressed cubics: 
$$z^3 = 3pz + 2q$$

$$(s+t)^{3} = 3p(s+t) + 2q$$

$$s^{3} + 3s^{2}t + 3st^{2} + t^{3} = 3p(s+t) + 2q$$

$$s^{3} + 3st(s+t) + t^{3} = 3p(s+t) + 2q$$

$$s^{3} + t^{3} = 2q \quad st = p \quad \Rightarrow t = p/s$$

$$s^{3} + p^{3}/s^{3} = 2q \quad (s^{3})^{2} - 2qs^{3} + p^{3} = 0$$

$$s^{3} = q \pm \sqrt{q^{2} - p^{3}} \quad t^{3} = q \mp \sqrt{q^{2} - p^{3}}$$

$$z = \sqrt[3]{q + \sqrt{q^{2} - p^{3}}} + \sqrt[3]{q - \sqrt{q^{2} - p^{3}}}$$

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• Rafaello Bombelli's example: 
$$z^3 = 15z + 4$$
 ( $p = 5, q = 2$ )  
 $z = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$   
 $z = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$  where  $i = \sqrt{-1}$   
( $a + ib$ )  $\pm (a' + ib') = (a \pm a') + i(b \pm b')$   
( $a + ib$ )( $a' + ib'$ ) =  $aa' + iab' + iba' + i^2bb'$   
= ( $aa' - bb'$ ) + i( $ab' + ba'$ )  
( $2 \pm i$ )<sup>3</sup> =  $8 \pm 12i + 6i^2 \pm i^3 = 8 \pm 12i - 6 \mp i = 2 \pm 11i$   
 $\therefore z = (2 + 11i) + (2 - 11i) = 4$ 

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• Quartics: Lodovico Ferrari (1540)

Cubics and quartics: Girolamo Cardano, Ars Magna (1545)

• Quintics: No general algebraic solution for degree  $\geq$  5.

(Paolo Ruffini 1799, Niels Abel 1824)

• Splitting:  $p(a) = 0 \Leftrightarrow p(X) = (X - a)q(X)$  for some q(X).

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Since deg r < deg(X - a) = 1, r is constant.

Plug in X = a to obtain r = 0.

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The largest m such that p(X) is a multiple of (X = a)<sup>m</sup> is called multiplicity.

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## Complex numbers

 $\mathbf{C} = \mathbf{R}[i] = \mathbf{R}[X] / < X^2 + 1 >$ 

- Multiples of  $X^2 + 1$  form a maximal ideal of the polynomial ring  $\mathbf{R}[X]$ . The factor ring **C** is a field.
- Cosets (shifts) of this principal ideal  $< X^2 + 1 >$  are equivalence classes, where two polynomials are considered equivalent when their difference is in the ideal, i.e. a multiple of  $X^2 + 1$ .
- Since  $X^2 \sim -1$ ,  $X^3 \sim -X$ ,  $X^4 \sim 1$ , etc., each coset has a unique representative of the form a + Xb. The coset is denoted by z = a + ib.
- $z \pm z' = (a + ib) \pm (a' + ib') = (a \pm a') + i(b \pm b')$  zz' = (a + ib)(a' + ib') = (aa' - bb') + i(ab' + ba')  $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$ where  $\overline{z}$  is the complex conjugate and |z| is the magnitude

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• Complex plane (Caspar Wessel 1799, Jean-Robert Argand 1806) Polar coordinates:  $z = a + ib = r(\cos \theta + i \sin \theta) = re^{i\theta}$ 

(Roger Cotes 1714, Leonhard Euler 1748)

$$e^{i\theta} = 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \dots = 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{i}{3!}\theta^3 + \dots$$
$$= (1 - \frac{1}{2!}\theta^2 + \dots) + i(\theta - \frac{1}{3!}\theta^3 + \dots) = \cos\theta + i\sin\theta$$

Our jewel. One of the most remarkable, almost astounding, formulas in all of mathematics.

- Richard Feynman.

• Linear algebra:  $\mathbf{C} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : [a, b] \in \mathbf{R}^2 \right\}$   $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a' & -b' \\ b' & a' \end{bmatrix} = \begin{bmatrix} aa' - bb' & -ab' - ba' \\ ba' + ab' & -bb' + aa' \end{bmatrix}$  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  (isotropic dilation + rotation) • Complex plane (Caspar Wessel 1799, Jean-Robert Argand 1806) Polar coordinates:  $z = a + ib = r(\cos \theta + i \sin \theta) = re^{i\theta}$ 

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### Complex multiplication

 $zz' = r(\cos \theta + i \sin \theta)r'(\cos \theta' + i \sin \theta')$ =  $(rr')[\cos \theta \cos \theta' - \sin \theta \sin \theta' + i(\sin \theta \cos \theta' + \cos \theta \sin \theta')]$ =  $(rr')[\cos(\theta + \theta') + i \sin(\theta + \theta')]$ =  $rr' = (rri\theta)(rt e^{i\theta'}) = (rrt)e^{i(\theta + \theta')}$ 

• 
$$ZZ' = (re^{i\theta})(r'e^{i\theta'}) = (rr')e^{i(\theta+\theta')}$$

Magnitudes multiply, phases add.

Complex powers f(z) = z<sup>n</sup>

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- Complex powers f(z) = z<sup>n</sup>
  - $x z^{2} = (x + iy)^{2} = (x^{2} y^{2}) + i(2xy)$  $x z^{2} = (x + iy)^{2} = (x^{2} - 3xy^{2}) + i(3x^{2}y - y^{2})$ 
    - $p \cdot z^n = (re^{i\theta})^n = r^n e^{in\theta}$  (trig version: Abraham de Moivre 1722)

- Complex multiplication
  - $ZZ' = r(\cos \theta + i \sin \theta)r'(\cos \theta' + i \sin \theta')$   $= (rr')[\cos \theta \cos \theta' - \sin \theta \sin \theta' + i(\sin \theta \cos \theta' + \cos \theta \sin \theta')]$   $= (rr')[\cos(\theta + \theta') + i \sin(\theta + \theta')]$ •  $ZZ' = (re^{i\theta})(r'e^{i\theta'}) = (rr')e^{i(\theta + \theta')}$ 
    - Magnitudes multiply, phases add.
  - Complex powers  $f(z) = z^n$ •  $z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$ •  $z^2 = (x + iy)^2 = (x^2 - 3xy^2) + i(3x^2y - y^2)$ •  $z^2 = (xe^{iy})^2 = x^2e^{iyt}$  (ing version: Abraham de Moivre 17) Phase oots multiplied by *n*.

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• Complex powers  $f(z) = z^n$ 

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$$z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$$

• 
$$z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

*z<sup>n</sup>* = (*re<sup>iθ</sup>*)<sup>*n*</sup> = *r<sup>n</sup>e<sup>inθ</sup>* (trig version: Abraham de Moivre 1722)
 Phase gets multiplied by *n*.

# Every non-constant complex polynomial has a root, and therefore, by induction, splits completely into linear factors.

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- Complex conjugation:  $a + ib \mapsto a ib$ The flip with respect to the real axis  $f(z) = \overline{z}$  is an automorphism of **C** keeping exactly **R** fixed.
- A real polynomial splits into linear factors and quadratics without real roots (i.e. with negative discriminant).
  - Suppose p(X) is a real polynomial. Conjugate p(z) = 0:
    - $0 = p(z) = a_0 + a_1 z + \dots a_n z^n = a_0 + a_1 \overline{z} + \dots a_n \overline{z}^n = p(\overline{z})$
    - Thus, complex roots come in conjugate pairs.
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#### Argument principle

The number of zeros (counted with multiplicities) of f(z) inside a loop is the winding number w of the image of the loop under f with respect to the origin.

(if *f* has poles, they need to be counted with negative multiplicities)

• An analytic approach to the winding number w• Complex logarithm (multivalued with period  $2\pi$ ) In  $z = \ln(re^{rt}) = \ln(e^{it} e^{rt}) = \ln e^{it} (rt^{rt}) = \ln r + 10$ (angle t is called the argument (phase) of z) • Integrate the logarithmic derivative of t around a loop  $\infty$  $\int_{0}^{1} \frac{d^{2}}{dt^{2}} dz = \int_{0}^{1} (\ln t(z))^{2} dz = \ln t(z)|_{0} = 2\pi t w$ 

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• Integrate the logarithmic derivative of f around a loop  $\gamma$ 

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = \int_{\gamma} (\ln f(z))' dz = \ln f(z)|_{\gamma} = 2\pi i w$$

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#### *n*-th roots

Special case:  $z^n = re^{i\theta}$ 

Let  $H = \{z \in \mathbb{C} : z^n = 1\}$ . *H* is a subgroup of the unit circle (which in turn is a subgroup of the multiplicative group of complex units  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ )

$$H = \left\{ e^{ikrac{2\pi}{n}} \colon k \in \mathbf{Z}_n 
ight\}$$
 — a regular *n*-gon.

The solution set is a coset of H

$$z = \sqrt[n]{r} e^{j\frac{\theta}{n}} H = \left\{ \sqrt[n]{r} e^{j\frac{\theta+2k\pi}{n}} \colon k \in \mathbf{Z}_n \right\}$$

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#### Discussion

Complaints > /dev/null

Ok, just kidding ...

gokhman@math.utsa.edu

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