Fundamental Theorem of Algebra

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Outline

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Introduction

No polynomial of the type $X^4 + a^4$ *(with* $a \neq 0$*) can be factored.*
— Gottfried Leibniz (1702) (he was wrong)

- Intersections of curves $\longrightarrow$ roots of polynomials
- Quadratics: $z^2 - 2mz + c = 0$

  $z^2 - 2mz + m^2 = m^2 - c \qquad (z-m)^2 = m^2 - c$

  $z = m \pm \sqrt{m^2 - c}$
- Cubics: $z^3 + 3az^2 + bz + c = 0$

  (Scipione del Ferro, Niccolò Fontana (Tartaglia), XVI)

  To eliminate z^2, shift the inflection point $z = -a$ to the origin:

  $(z-a)^3 + 3a(z-a)^2 + b(z-a) + c = 0$

  $z^3 - 3z^2a + 3za^2 - a^3 + 3a(z^2 - 2za + a^2) + b(z-a) + c = 0$

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- Depressed cubics: $z^3 = 3pz + 2q$

$(s+t)^3 = 3p(s+t) + 2q$

$s^3 + 3s^2t + 3st^2 + t^3 = 3p(s+t) + 2q$

$s^3 + 3st(s+t) + t^3 = 3p(s+t) + 2q$

$s^3 + t^3 = 2q \qquad st = p \qquad \Rightarrow t = p/s$

$s^3 + p^3/s^3 = 2q \qquad (s^3)^2 - 2qs^3 + p^3 = 0$

$s^3 = q \pm \sqrt{q^2 - p^3} \qquad t^3 = q \mp \sqrt{q^2 - p^3}$

$z = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$

- Depressed cubics: $z^3 = 3pz + 2q$
$z = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$
- Rafaello Bombelli's example: $z^3 = 15z + 4 \quad (p = 5, q = 2)$
$z = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$
$z = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \qquad \text{where } i = \sqrt{-1}$
$(a + ib) \pm (a' + ib') = (a \pm a') + i(b \pm b')$
$(a + ib)(a' + ib') = aa' + iab' + iba' + i^2 bb'$
$= (aa' - bb') + i(ab' + ba')$
$(2 \pm i)^3 = 8 \pm 12i + 6i^2 \pm i^3 = 8 \pm 12i - 6 \mp i = 2 \pm 11i$
$\therefore z = (2 + 11i) + (2 - 11i) = 4$

- Higher degree
  - Quartics: Lodovico Ferrari (1540)

    Cubics and quartics: Girolamo Cardano, *Ars Magna* (1545)
  - Quintics: No general algebraic solution for degree ≥ 5.

    (Paolo Ruffini 1799, Niels Abel 1824)
- Splitting: $p(a) = 0 \Leftrightarrow p(X) = (X - a)q(X)$ for some $q(X)$.
  - By long division $p(X) = (X - a)q(X) + r$.

    Since $\deg r < deg(X - a) = 1$, r is constant.
  - Plug in $X = a$ to obtain $r = 0$.
  - The largest m such that $p(X)$ is a multiple of $(X - a)^m$ is called multiplicity.

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Complex numbers

$\mathbf{C} = \mathbf{R}[i] = \mathbf{R}[X]/ < X^2 + 1 >$

- Multiples of $X^2 + 1$ form a maximal ideal of the polynomial ring $\mathbf{R}[X]$. The factor ring $\mathbf{C}$ is a field.
- Cosets (shifts) of this principal ideal $< X^2 + 1 >$ are equivalence classes, where two polynomials are considered equivalent when their difference is in the ideal, i.e. a multiple of $X^2 + 1$.
- Since $X^2 \sim -1$, $X^3 \sim -X$, $X^4 \sim 1$, etc., each coset has a unique representative of the form $a + Xb$. The coset is denoted by $z = a + ib$.
- $z \pm z' = (a + ib) \pm (a' + ib') = (a \pm a') + i(b \pm b')$
  $zz' = (a + ib)(a' + ib') = (aa' - bb') + i(ab' + ba')$
  $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$
  where $\bar{z}$ is the complex conjugate and $|z|$ is the magnitude.

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- Complex plane (Caspar Wessel 1799, Jean-Robert Argand 1806)

  Polar coordinates: $z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$

  (Roger Cotes 1714, Leonhard Euler 1748)

  $e^{i\theta} = 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \ldots = 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{i}{3!}\theta^3 + \ldots$

  $= (1 - \frac{1}{2!}\theta^2 + \ldots) + i(\theta - \frac{1}{3!}\theta^3 + \ldots) = cos\theta + i\sin\theta$

  Our jewel. One of the most remarkable, almost astounding, formulas in all of mathematics.

  — Richard Feynman.

- Linear algebra: $\mathbf{C} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : [a, b] \in \mathbf{R}^2 \right\}$

  $$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a' & -b' \\ b' & a' \end{bmatrix} = \begin{bmatrix} aa' - bb' & -ab' - ba' \\ ba' + ab' & -bb' + aa' \end{bmatrix}$$

  $$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ (isotropic dilation + rotation)

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- Complex multiplication
  - $zz' = r(\cos\theta + i\sin\theta)r'(\cos\theta' + i\sin\theta')$
    $= (rr')[\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')]$
    $= (rr')[\cos(\theta + \theta') + i\sin(\theta + \theta')]$
  - $zz' = (re^{i\theta})(r'e^{i\theta'}) = (rr')e^{i(\theta+\theta')}$
    Magnitudes multiply, phases add.
- Complex powers $f(z) = z^n$
  - $z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$
  - $z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$
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    Phase gets multiplied by n.

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Every non-constant complex polynomial has a root, and therefore, by induction, splits completely into linear factors.

- Early attempts, assuming existence (incomplete ☹):
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Fundamental Theorem of Algebra

Every non-constant complex polynomial has a root, and therefore, by induction, splits completely into linear factors.

- Let $p(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n$ belong to $\mathbf{C}[z]$. Assume p is never zero. Then $a_0 \neq 0$ (otherwise $p(0) = 0$).
- For each r the image of the circle $\{z = re^{i\theta} \colon -\pi < \theta \leq \pi\}$ is a loop in $\mathbf{C} \setminus \{0\}$. Let $\varphi(r)$ be the winding number w of this loop around the origin. Since φ is continuous and its image is discrete, φ is constant (you can't change the winding number without crossing the origin).
- For small $|z| = r$, we have $p(z) \approx a_0$, so $\varphi(r) = 0$.
- For large r, the dominant term in $p(z)$ is z^n, so $\varphi(r) = n$.
- Since φ is constant, $n = 0$. $\smile$

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- Complex conjugation: $a + ib \quad \mapsto \quad a - ib$
  The flip with respect to the real axis $f(z) = \bar{z}$ is an automorphism of **C** keeping exactly **R** fixed.
- A real polynomial splits into linear factors and quadratics without real roots (i.e. with negative discriminant).
  - Suppose $p(X)$ is a real polynomial. Conjugate $p(z) = 0$:
    $0 = \overline{p(z)} = \overline{a_0 + a_1 z + \dots a_n z^n} = a_0 + a_1 \bar{z} + \dots a_n \bar{z}^n = p(\bar{z})$
    Thus, complex roots come in conjugate pairs.
  - $(X - (a + ib))(X - (a - ib)) = (X - a)^2 + b^2$

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Argument principle

The number of zeros (counted with multiplicities) of $f(z)$ inside a loop is the winding number w of the image of the loop under f with respect to the origin.

(if f has poles, they need to be counted with negative mutliplicities)

- An analytic approach to the winding number w
  - Complex logarithm (multivalued with period $2\pi i$)

    $$\ln z = \ln(re^{i\theta}) = \ln(e^{\ln r}e^{i\theta}) = \ln e^{\ln r + i\theta} = \ln r + i\theta$$

    (angle θ is called the argument (phase) of z)
  - Integrate the logarithmic derivative of f around a loop γ

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma (\ln f(z))'\,dz = \ln f(z)\big|_\gamma = 2\pi i\, w$$

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n-th roots

Special case: $z^n = re^{i\theta}$

Let $H = \{z \in \mathbf{C} \colon z^n = 1\}$. H is a subgroup of the unit circle (which in turn is a subgroup of the multiplicative group of complex units $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$)

$H = \left\{ e^{ik\frac{2\pi}{n}} : k \in \mathbf{Z}_n \right\}$ — a regular n-gon.

The solution set is a coset of H

$$z = \sqrt[n]{r} e^{i\frac{\theta}{n}} H = \left\{ \sqrt[n]{r} e^{i\frac{\theta + 2k\pi}{n}} : k \in \mathbf{Z}_n \right\}$$

Discussion

Complaints $>$ /dev/null

Ok, just kidding ...

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