Exterior product and differentiation.

We will look at the algebra of differentials and the exterior derivative, a natural generalization of the usual derivative.

In the early days calculus was done mainly with differentials, not derivatives, with the following rules $d(c) = 0$, $d(f+g) = df + dg$, $d(fg) = df\, g + f\, dg$, etc. We see vestiges of this in modern calculus texts typically in sections on linear approximation and in integration by substitution. For functions of one variable the differential is related to the derivative by $df = f'dx$, so to save on writing dx a lot, most calculus books have dropped it and concentrate mostly on f'. When one gets to functions of several variables, however, it becomes clear that we should have kept the dx, just as we teach kids to brush their teeth even though the first set will fall out anyway.

Like his contemporaries Leonhard Euler (1707–1783) consistenly used differentials, but was stumped by the problem of substitutions in multiple integrals. The problem was solved by Hermann Grassmann (1809–1877) and Élie Cartan (1869–1951), who discovered that multiplication of differentials (the wedge $\wedge$) is alternating in the sense that switching variables causes a sign change. A big clue was the fact that the area of a parallelogram formed by two vectors in the plane, or a parallelepiped formed by three vectors in 3-space is an alternating function (the determinant). In fact, we will see that the famous vector products are special cases of the wedge product.

The exterior derivative generalizes the notion of the derivative. Its special cases include the gradient, curl and divergence. The notion of derivative is a local one, so we will start by looking at a neighborhood U of a fixed point p.

1 Tangent space

Given a open set U in n-dimensional Euclidean space the tangent space to U at a point $p \in U$ is the n-dimensional Euclidean space with origin at p and is denoted $T_p(U)$.

You can think of $T_p(U)$ as the space of all direction vectors from p. Sometimes $T_p(U)$ is identified with the space of all directional derivatives at p. Given a direction vector $u \in T_p(U)$, the directional derivative of a function $f : U \to \mathbf{R}$ is

$$\frac{d}{dt} f(p + tu).$$

If U comes equipped with cartesian coordinates $x_1, x_2, ...x_n$ we then have corresponding coordinates for $T_p(U)$, namely $\Delta x_i = x_i - p_i$. In view of the identification of $T_p(U)$ with the space of all directional derivatives Δx_i are sometimes denoted by $\partial/\partial x_i$.

2 The space of differentials

The space of differentials $T_p^*(U)$ is the dual vector space of the tangent space, i.e. the vector space of linear maps $T_p(U) \to \mathbf{R}$.

The dual cartesian coordinate basis for $T_p^*(U)$ is denoted by dx_i. Recall the definition of dual basis $dx_i(\Delta x_j) = \delta_{ij}$. You can think of dx_i as the projection to the i-th coordinate, i.e. given a vector $u \in T_p(U)$, we define $dx_i(u) = u_i$.

3 Tensor powers

Given a vector space V we can construct its tensor power $\bigotimes^k V = V \otimes V \otimes ...V$ as follows. We take the cartesian power $\prod^k V = V \times V \times ...V$ and consider the vector space W spanned by the elements of $\prod^k V$ (considered as a set). Then we take the subspace of W generated by the multilinear relations (e.g. elements of the form $(u, v) + (u, w) - (u, v + w)$, $a(u, v) - (au, v)$ etc. when $k = 2$), and factor it out to obtain $\bigotimes^k V$.

The motivation for this comes from multilinear maps, i.e. maps linear in each variable. The set of all multilinear maps $\prod^k V \to \mathbf{R}$ is naturally equivalent to the set of linear maps $\bigotimes^k V \to \mathbf{R}$. This is known as the universal property of multilinear maps. Note that $(\bigotimes^k V)^* = \bigotimes^k V^*$.

Given a basis $\{v_1, v_2, ...v_n\}$ for V we have a basis for $\bigotimes^k V$ which consists of all distinct sequences of v_i of length k (all distinct k-tuples of v_i). Such sequences are typically written with $\otimes$ inbetween.

Example 3.1 If $\{x, y\}$ is a basis for V, then $\{x \otimes x, x \otimes y, y \otimes x, y \otimes y\}$ is a basis for $V \otimes V$.

In particular, for $k \leq n$

$$\dim \bigotimes^k V = n^k$$

The various tensor powers can be combined in a single graded algebra

$$\bigoplus_{k=1}^{\infty} \bigotimes^k V = V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus ...$$

4 Exterior powers

If in the above construction of tensor powers we include the alternating relations (e.g. $(u, v, w) + (v, u, w)$, etc. when $k = 3$), we obtain the exterior power $\bigwedge^k V$.

The motivation for this comes from alternating multilinear maps, i.e. maps linear in each variable that change sign if two variables are transposed (more generally, for any permutation of the variable the sign changes according to the parity of the permutation). The set of all alternating multilinear maps $\prod^k V \to \mathbf{R}$ is naturally equivalent to the set of linear maps $\bigwedge^k V \to \mathbf{R}$. This is known as the universal property of alternating multilinear maps. Note that $(\bigwedge^k V)^* = \bigwedge^k V^*$.

Given a basis $\{v_1, v_2, ...v_n\}$ for V we have a basis for $\bigwedge^k V$ which consists of all distinct sequences of v_i of length k, where we require that in any given sequence the v_i are distinct and arranged in a particular order (e.g. the order of increasing index). Such sequences are typically written with $\wedge$ inbetween.

Example 4.1 If $\{x, y, z\}$ is a basis for V, then $\{y \wedge z, z \wedge x, x \wedge y\}$ is a basis for $V \wedge V$.

In particular, $\bigwedge^k V = 0$ for $k > n$ and for $k \leq n$

$$\dim \textstyle\bigwedge^k V = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example 4.2 Let $A: V \to V$ be a linear transformation. Take its n-th exterior power $\alpha\colon \prod^n V \to \bigwedge^n V$ by letting $\alpha(u_1, u_2, ...u_n) = Au_1 \wedge Au_2 \wedge ...Au_n$. This is an alternating multilinear map so by the universal property we obtain a corresponding linear transformation of $\bigwedge^n V$. The latter vector space is one dimensional, so the transformation is multiplication by a scalar. It can be seen without difficulty that if A is represented by a matrix (a_{ij}) with respect to any basis of V, then the above scalar is the determinant of that matrix, i.e.

$$Au_1 \wedge Au_2 \wedge ...Au_n = |A|\, u_1 \wedge u_2 \wedge ...u_n, \quad \text{where } |A| = \sum_{\pi \in \Sigma_n} \operatorname{sgn} \pi \prod_{i=1}^{n} a_{i\pi(i)}.$$

The various exterior powers can be combined in a single graded algebra

$$\bigoplus_{k=1}^{\infty} \textstyle\bigwedge^k V = \displaystyle\bigoplus_{k=1}^{n} \textstyle\bigwedge^k V = V \oplus (V \wedge V) \oplus (V \wedge V \wedge V) \oplus ...(\bigwedge^n V)$$

Unlike the tensor algebra, this is finite dimensional as a vector space.

$u \wedge v = (u_x dx + u_y dy + u_z dz) \wedge (v_x dy \wedge dz + v_y dz \wedge dx + v_z dx \wedge dy)$ $= (u_x v_x + u_y v_y + u_z v_z) dx \wedge dy \wedge dz$	$u \cdot v$ dot product
$u \wedge v = (u_x dx + u_y dy + u_z dz) \wedge (v_x dx + v_y dy + v_z dz)$ $= (u_y v_z - u_z v_y) dy \wedge dz + (u_z v_x - u_x v_z) dz \wedge dx + (u_x v_y - u_y v_x) dx \wedge dy$ $= \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} dy \wedge dz + \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix} dz \wedge dx + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} dx \wedge dy$	$u \times v$ cross product
$u \wedge v \wedge w = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} dx \wedge dy \wedge dz$ $= \left(u_x \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} + u_y \begin{vmatrix} v_z & v_x \\ w_z & w_x \end{vmatrix} + u_z \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} \right) dx \wedge dy \wedge dz$	det (triple prod) $(uvw) = u \cdot (v \times w)$

5 Exterior algebra of the space of differentials

The exterior powers of the space of differentials $\bigwedge^k T_p^*(U)$ can be thought of as the vector spaces of multilinear alternating maps $\prod^k T_p(U) \to \mathbf{R}$.

6 Differential forms

Differential 0-forms are smooth maps $U \to \mathbf{R}$. Differential 1-forms are smooth maps $U \to T_p^*(U)$ taking $p \in U$ to an element of $T_p^*(U)$. Differential k-forms are smooth maps $U \to \bigwedge^k T_p^*(U)$ taking $p \in U$ to an element of $\bigwedge^k T_p^*(U)$. The set of k-forms on U will be denoted by $\Omega^k(U) = \bigwedge^k \Omega^1(U)$.

degree	name	cartesian coordinate form	dim $\bigwedge^k T_p^*(\mathbf{R}^3)$
0-form	scalar form	$F = F(x, y, z)$	1
1-form	work form	$\omega = Adx + Bdy + Cdz$	3
2-form	flux form	$\varphi = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$	3
3-form	density form	$\rho = Fdx \wedge dy \wedge dz$	1

7 Exterior derivative

Given a smooth function $F\colon U \to \mathbf{R}^m$, its differential dF is a linear map that approximates F near p. We can think of dF as a map $T_p(U) \to \mathbf{R}^m$.

In cartesian coordinates we have

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i = DF \cdot dx,$$

where DF is the matrix of partial derivatives of components of F known as the Jacobian matrix.

We generalize d to the graded algebra of differentials by constructing linear maps $d\colon \Omega^k(U) \to \Omega^{k+1}(U)$ (all called d by an abuse of notation) such that

$$d \circ d = 0$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$$

It is intuitively clear how armed with these rules one can compute d of any form. The first rule is known as the first part of the Poincaré lemma and can be formulated as in terms of the equality of mixed partial derivatives. The second rule is a generalization of the product rule of differentiation (sometimes known as the Leibniz rule).

Here we show the vector forms of exterior differentiation (see Darling) (here we use the symbolic notation $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, where ∇ is thought of as a vector differential operator obeying the usual algebraic rules, but with the partial derivatives applied rather than multiplied):

exterior derivative	vector interpretation
$dF = \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy + \dfrac{\partial F}{\partial z}dz$	$\operatorname{grad} F = DF = \nabla F = \left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}\right)$
$d\omega = d(Adx + Bdy + Cdz) = \left(\dfrac{\partial C}{\partial y} - \dfrac{\partial B}{\partial z}\right) dy \wedge dz$ $+ \left(\dfrac{\partial A}{\partial z} - \dfrac{\partial C}{\partial x}\right) dz \wedge dx + \left(\dfrac{\partial B}{\partial x} - \dfrac{\partial A}{\partial y}\right) dx \wedge dy$	$\operatorname{curl} \Psi = \operatorname{rot} \Psi = \nabla \times \Psi$ $= \left(\dfrac{\partial \Psi_z}{\partial y} - \dfrac{\partial \Psi_y}{\partial z}, \dfrac{\partial \Psi_x}{\partial z} - \dfrac{\partial \Psi_z}{\partial x}, \dfrac{\partial \Psi_y}{\partial x} - \dfrac{\partial \Psi_x}{\partial y}\right)$
$d\varphi = d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy)$ $= \left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz$	$\operatorname{div} \Phi = \nabla \cdot \Phi = \dfrac{\partial \Phi_x}{\partial x} + \dfrac{\partial \Phi_y}{\partial y} + \dfrac{\partial \Phi_z}{\partial z}$

Vector forms of the two rules of exterior differentiation (see Gradshteyn/Ryzhik 10.31):

$d \circ d = 0$	
$\operatorname{curl}(\operatorname{grad} F) = 0$	$\nabla \times (\nabla F) = 0$
$\operatorname{div}(\operatorname{curl} \Phi) = 0$	$\nabla \cdot (\nabla \times \Phi) = 0$

$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$	
$\operatorname{grad}(FG) = (\operatorname{grad} F)G + F(\operatorname{grad} G)$	$\nabla(FG) = (\nabla F)G + F(\nabla G)$
$\operatorname{curl}(F\Psi) = (\operatorname{grad} F) \times \Psi + F(\operatorname{curl} \Psi)$	$\nabla \times (F\Psi) = (\nabla F) \times \Psi + F(\nabla \times \Psi)$
$\operatorname{div}(\Psi_1 \times \Psi_2) = (\operatorname{curl} \Psi_1) \cdot \Psi_2 - \Psi_1 \cdot (\operatorname{curl} \Psi_2)$	$\nabla \cdot (\Psi_1 \times \Psi_2) = (\nabla \times \Psi_1) \cdot \Psi_2 - \Psi_1 \cdot (\nabla \times \Psi_2)$
$\operatorname{div}(F\Phi) = (\operatorname{grad} F) \cdot \Phi + F(\operatorname{div} \Phi)$	$\nabla \cdot (F\Phi) = (\nabla F) \cdot \Phi + F(\nabla \cdot \Phi)$