Holomorphic functions

Liouville's Theorem: If $f \in \mathcal{H}(\mathbf{C})$ is bounded, then f is constant.
Proof: Cauchy's Integral Formula with $k > 0$ over a circle of radius r at $0 \Rightarrow |c_k| \le \sup|f(z)|/r^k \to 0$ as $r \to \infty$.

Fundamental Theorem of Algebra (K.F. Gauss)**:** If $f \in \mathbf{C}[X] \subset \mathcal{H}(\mathbf{C})$ is not constant, then $V(f) \ne \emptyset$.
Proof: Suppose p has no zeros. Since $p(z) \to \infty$ as $z \to \infty$, $1/p \in \mathcal{H}(\mathbf{C})$ is bounded.

Analytic continuation principle: If $\Omega \subseteq \mathbf{C}$ is a domain and $f \in \mathcal{H}(\Omega)$, then $V(f) \stackrel{\text{def}}{=} \{z \in \Omega\colon f(z) = 0\}$ is discrete or $= \Omega$.
Proof: Since f is continuous, $V(f)$ is closed in Ω. Suppose $f(z_0) = 0$ and $f \not\equiv 0$ in any neighborhood of z_0. Expanding f in a Taylor series at z_0 and factoring out the maximum power of $z - z_0$ we can write $f(z) = (z - z_0)^n g(z)$, where $g(z_0) \ne 0$. Since g is continuous, $g \ne 0$ in a neighborhood of z_0, so z_0 is not a limit point of $V(f)$. Let $U = \{z \in \Omega\colon f \equiv 0 \text{ in some neighborhood of } z\}$. Then U is closed and open in Ω, which is connected, so $U = \emptyset$ or $U = \Omega$.

Corollary: $\mathcal{H}(\Omega)$ is an integral domain.
Proof: Let $f, g \in \mathcal{H}(\Omega)$. If $fg \equiv 0$, then $V(fg) = V(f) \bigcup V(g) = \Omega$. Since $V(f)$, $V(g)$ are not both discrete, $f \equiv 0$ or $g \equiv 0$.

Open Mapping Theorem: If $f \in \mathcal{H}(\Omega)$ is not constant, then f is open (takes open sets to open sets).
Proof: Assume $0 \in \Omega$, $f(0) = 0$ and let D be a disk at 0 with $\rho = \min\limits_{\partial D} |f| > 0$. If $B_\rho(0) \not\subseteq f(\Omega)$, then $B_{\rho/2}(0) \subseteq f(\Omega)$. Indeed, if $|w| < \rho$ and $f(w) \notin f(\Omega)$, then $g(z) = \dfrac{1}{f(z) - w} \in \mathcal{H}(\Omega)$, so $|g(0)| = \dfrac{1}{|w|} \le \sup\limits_{\partial D} \dfrac{1}{|f(z) - w|} \le \dfrac{1}{\rho - |w|}$ and $|w| \ge \rho/2$.

Maximum Modulus Principle: If $f \in \mathcal{H}(\Omega)$ is not constant, then $|f|$ does not attain a maximum in Ω.
Proof: Modulus $\Omega \to [0, \infty)$ is an open map, so $|f|$ is an open map. Thus, $|f(\Omega)| \subseteq [0, \infty)$ is open and has no maximum.

Laurent Series: If Ω is an annulus at z_0 and $f \in \mathcal{H}(\Omega)$, then $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$ with $c_k = \dfrac{1}{2\pi i} \displaystyle\int_L (w - z_0)^{-k-1} f(w)\, dw$.
Proof: Let $z \in \Omega$ and L_0 and L_1 two circles in Ω around z_0 with z inside L_1 and outside L_0. Since $L_1 - L_0$ is a boundary, Cauchy's integral formula gives $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{L_1 - L_0} \dfrac{f(w)}{w - z}\, dw = \dfrac{1}{2\pi i} \left(\int_{L_1} \dfrac{f(w)}{w - z}\, dw - \int_{L_0} \dfrac{f(w)}{w - z}\, dw \right)$, but

$$\int_{L_1} \frac{f(w)\, dw}{w - z} = \int_{L_1} \frac{1}{1 - \left(\dfrac{z - z_0}{w - z_0}\right)} \frac{f(w)\, dw}{(w - z_0)} = \int_{L_1} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^k \frac{f(w)\, dw}{(w - z_0)} = \sum_{k=0}^{\infty} \left(\int_{L_1} \frac{f(w)\, dw}{(w - z_0)^{k+1}} \right) (z - z_0)^k \text{ and}$$

$$\int_{L_0} \frac{f(w)\, dw}{w - z} = -\int_{L_0} \frac{1}{1 - \left(\dfrac{w - z_0}{z - z_0}\right)} \frac{f(w)\, dw}{(z - z_0)} = -\int_{L_0} \sum_{k=0}^{\infty} \left(\frac{w - z_0}{z - z_0}\right)^k \frac{f(w)\, dw}{(z - z_0)} = -\sum_{k=0}^{\infty} \left(\int_{L_0} (w - z_0)^k f(w)\, dw \right) (z - z_0)^{-k-1}.$$

Note: If $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$ in an annulus Ω and L is a circle in Ω, then by termwise integration $\displaystyle\int_L f(z)\, dz = 2\pi i\, c_{-1}$.

Singularities: $z_0 \in \Omega$ is a *singularity* of f if $f \in \mathcal{H}(U \setminus \{z_0\})$ for a disc U at z_0. If $f \not\equiv 0$, not all $c_k = 0$ in the Laurent expansion of f in the annulus $U \setminus \{z_0\}$. Let $\operatorname{ord}_{z_0} f = \inf\{k\colon c_k \ne 0\}$. A point z is an *essential* singularity of f when $\operatorname{ord}_z f = -\infty$, a *pole* of multiplicity n when $\operatorname{ord}_z f = -n < 0$, a *removable* singularity when $\operatorname{ord}_z f \ge 0$, and a zero of multiplicity n when $\operatorname{ord}_z f = n > 0$.

Riemann Extension Theorem: Let $z_0 \in \Omega$. If $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ is bounded, then we can extend $f \in \mathcal{H}(\Omega)$.
Proof: Let L be a circle around z_0 of radius r and $M = \sup|f(z)|$. Since $|c_k| \le 2\pi r^{-k} M$, $c_k = 0$ for $k < 0$.

Meromorphic functions: A function is called *meromorphic* ($h \in \mathcal{M}(\Omega)$) when $h \in \mathcal{H}(\Omega \setminus S)$ and h is in the field of fractions of $\mathcal{H}(U)$ for all sufficiently small neighborhoods $U \subseteq \Omega$.

Theorem: $h \in \mathcal{M}(\Omega) \Leftrightarrow \exists$ discrete $S \subset \Omega$ with $h \in \mathcal{H}(\Omega \setminus S)$ and points of S are not essential singularities of h.
Proof: Let $h = f/g$ with $f, g \in \mathcal{H}(\Omega)$ and $g \not\equiv 0$. Then $h \in \mathcal{H}(\Omega \setminus V(g))$. Let $z_0 \in V(g)$ and expand f and g in Taylor series $f(z) = \displaystyle\sum_{k=0}^{\infty} a_k (z - z_0)^k$, $g(z) = \displaystyle\sum_{k=0}^{\infty} b_k (z - z_0)^k$. Let $g(z) = (z - z_0)^n w(z)$, where $w(z_0) \ne 0$. Then f/w is holomorphic in a neighborhood of z_0, so $f(z)/w(z) = \displaystyle\sum_{k=0}^{\infty} c_k (z - z_0)^k$ and $f(z)/g(z) = \displaystyle\sum_{k=0}^{\infty} c_k (z - z_0)^{k-n} = \sum_{k=-n}^{\infty} c_{k+n} (z - z_0)^k$. Conversely, let $h \in \mathcal{H}(\Omega \setminus S)$ and $z_0 \in S$. Expand h in a Laurent series $h(z) = \displaystyle\sum_{k=n}^{\infty} c_k (z - z_0)^k$. Then $h(z) = \displaystyle\sum_{k=0}^{\infty} c_{k-n} (z - z_0)^k / (z - z_0)^n$.

Theorem: If $z \in \Omega$, $\operatorname{ord}_z\colon \mathcal{M}(\Omega) \setminus \{0\} \to \mathbf{Z}$ is a valuation, i.e. $\operatorname{ord}_z(fg) = \operatorname{ord}_z f + \operatorname{ord}_z g$ and $\operatorname{ord}_z(f + g) \ge \min\{\operatorname{ord}_z f, \operatorname{ord}_z g\}$.

Value distribution: The behavior of a meromorphic function h in a neighborhood of a singulariy z_0 is fairly simple: either z_0 is removable so $\lim\limits_{z \to z_0} h(z) = h(z_0)$ or $\lim\limits_{z \to z_0} h(z) = \infty$. If z_0 is an essential singularity of $f \in \mathcal{H}(\Omega \setminus \{z_0\})$, then the complement of $f(\Omega)$ is a singleton or empty (E. Picard).