NATURAL TRANSFORMATIONS AND ADJOINTS

1 natural transformations and $[-,-]$

Categories and functors form a metacategory $\mathcal{M}$. For a category $\mathcal{C}$ we assume that $\mathrm{Obj}(\mathcal{C})$ is a proper class and for any $a, b \in \mathrm{Obj}(\mathcal{C})$ $\mathcal{C}[a,b]$ is a set. If $\mathrm{Obj}(\mathcal{C})$ is a set, then $\mathcal{C}$ is called a small category. The category of small categories is denoted by CAT. For $\mathcal{C} \in \mathrm{Obj}(\mathrm{CAT})$ we have a functor $\mathcal{C}[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{SET}$ taking a pair of objects in $\mathcal{C}$ to the set of morphisms between them. The product of categories takes place in $\mathcal{M}$ and is the obvious thing. A morphism in $\mathcal{C}^{op} \times \mathcal{C}[(a_1,b_1),(a_2,b_2)]$ is a pair of morphisms $\varphi\colon a_2 \to a_1$, $\psi\colon b_1 \to b_2$ in $\mathcal{C}$. Its value is the function $\mathcal{C}[\varphi,\psi]\colon \mathcal{C}[a_1,b_1] \to \mathcal{C}[a_2,b_2]$ taking $\gamma\colon a_1 \to b_1$ to $\psi \circ \gamma \circ \varphi$.

Definition 1.1 *Suppose $\mathcal{A}, \mathcal{B}$ are categories and $F, G\colon \mathcal{A} \to \mathcal{B}$ are functors. A natural transformation $\theta\colon F \to G$ is a collection of morphisms $\{\theta(a)\colon F(a) \to G(a) : a \in \mathrm{Obj}(\mathcal{A})\}$ such that for all morphisms $\varphi\colon a_1 \to a_2 \in \mathcal{A}$ we have $\theta(a_2) \circ F(\varphi) = G(\varphi) \circ \theta(a_1)$. If the morphisms $\theta(a)$ are all isomorphisms, then θ is called a natural equivalence.*

Composition of natural transformations works in the obvious way and is associative. In fact, if $\mathcal{J}$ is a small category and $\mathcal{A}$ is a category, then $\mathcal{M}[\mathcal{J},\mathcal{A}]$ and the collection of all natural transformations between these functors form the objects and morphisms respectively of a category denoted by $\mathcal{A}^{\mathcal{J}}$. If $\mathcal{A}$ is also small, then so is $\mathcal{A}^{\mathcal{J}}$. We have the diagonal functor $\Delta\colon \mathcal{A} \to \mathcal{A}^{\mathcal{J}}$ taking an object to the corresponding constant functor.

Example 1.1 A morphism is natural means that we are really talking about a whole collection of them (one for each object) forming a natural transformation. For example the natural homomorphism from a group G to $G/[G,G]$ can be thought of as a natural transformation from the identity functor on the category of groups to the abelianization functor $Ab\colon \mathrm{GPS} \to \mathrm{GPS}$.

Given $a \in \mathrm{Obj}(\mathcal{C})$ we have the functor $\mathcal{C}[a,-]\colon \mathcal{C} \to \mathrm{SET}$. A functor naturally equivalent to such a functor is called representable. Given a functor $F : \mathcal{C} \to \mathrm{SET}$, to $\gamma \in F(a)$ we assign a natural transformation $E_\gamma : \mathcal{C}[a,-] \to F$ defined as follows: if $x \in \mathrm{Obj}(\mathcal{C})$, then $\varphi \in \mathcal{C}[a,x]$ goes to $F(\varphi)(\gamma) \in F(x)$. Conversely, a natural transformation $E\colon \mathcal{C}[a,-] \to F$ is determined by $E(1_a)$. We have $E = E_{E(1_a)}$. This is the Yoneda lemma.

2 adjoints

Definition 2.1 *Given $\mathcal{X}, \mathcal{A} \in \mathrm{Obj}(\mathrm{CAT})$, $F \in \mathcal{A}^{\mathcal{X}}, G \in \mathcal{X}^{\mathcal{A}}$ we get two functors $\mathcal{A}[F(-),-]$, $\mathcal{X}[-,G(-)] : \mathcal{X}^{op} \times \mathcal{A} \to \mathrm{SET}$. If there is a natural equivalence between them, it is called an adjunction. The functor F is called the left adjoint of G and G is called the right adjoint of F.*

We will give three other equivalent formulations of adjunction:

- ☐ Let $b \in \mathrm{Obj}(\mathcal{X})$. Then $\mathcal{A}[F(b), F(b)] \simeq \mathcal{X}[b, GF(b)]$. Let η_b denote the image of $1_{F(b)}$ under this isomorphism of sets. The collection of η_b forms a natural transformation $\eta : I_{\mathcal{X}} \to GF$ and is called the unit of the adjunction. The unit has the property that if $\varphi \in \mathcal{X}[b, G(a)]$, then $\exists! \psi \in \mathcal{A}[F(b), a]$ such that $G(\psi) \circ \eta_b = \varphi$. Conversely, given a natural transformation $\eta\colon I_{\mathcal{X}} \to GF$ satisfying the above property, we can reconstruct the adjunction by taking a $\gamma \in \mathcal{A}[F(b), a]$ to $G(\gamma) \circ \eta_b$.

- ☐ In a similar vein, if $a \in \mathrm{Obj}(\mathcal{A})$, we have $\mathcal{A}[FG(a), a] \simeq \mathcal{X}[G(a), G(a)]$. The inverse image of $1_{G(a)}$ under this bijection is denoted by ϵ_a and these form a functor $\epsilon\colon FG \to I_{\mathcal{A}}$ called the counit of the adjunction. The counit has the property that if $\varphi \in \mathcal{A}[F(b), a]$, then $\exists! \psi \in \mathcal{X}[b, G(a)]$ such that $\epsilon_a \circ F(\psi) = \varphi$. Again, given a natural transformation $\epsilon : FG \to I_{\mathcal{A}}$ satisfying the above property, we can reconstruct the adjunction by taking a $\gamma \in \mathcal{X}[b, G(a)]$ to $\epsilon_a \circ F(\gamma)$.

- ☐ Finally we get the triangle identities $G(\epsilon_a) \circ \eta_{G(a)} = I_{G(a)}$ and $\epsilon_{F(b)} \circ F(\eta_a) = I_{F(b)}$. The adjunction can be reconstructed from these.

Example 2.1 If $\mathcal{A}$ is a category, the left and right adjoints of the diagonal functor $\Delta : \mathcal{A} \to \mathcal{A}^{\mathcal{J}}$ are famous universal constructions for various choices of a small category $\mathcal{J}$. For example if $\mathcal{J}$ consists of two objects and only the identitity morphisms, a functor in $\mathcal{A}^{\mathcal{J}}$ is determined by an ordered pair of objects in $\mathcal{A}$. Here the right adjoint of Δ is the product and the left adjoint is the coproduct. In SET for the product the unit is the diagonal map and for the coproduct (disjoint union) the counit is the fold map. By allowing more objects and morphisms in $\mathcal{J}$ we get the usual pullbacks, pushouts, equalizers, limits etc.

Adjoints are unique up to natural equivalence. Adjoint pairs can be composed (associative). Left adjoints preserve coproducts, pushouts and direct limits (colimits). Right adjoints preserve products, pullbacks and inverse limits (limits). The adjoint functor theorem says that if $\mathcal{A}$ has pullbacks and arbitrary products and $G\colon A \to X$ preserves them in addition to satisfying a so-called solution set condition, then one can construct a left adjoint for G.

3 examples of adjoints

Inclusion functors occur whenever we have a full subcategory and many of them are forgetful. They provide a lot of the examples of right adjoints.

- □ A concrete category is one whose objects and morphisms are sets and functions. The forgetful functor from a concrete category to SET is the right adjoint for the free object functor. The unit is formed by the canonical inclusion maps from sets to the free objects they generate.
  - ▹ A free semigroup on a set X is the collection of words of the elements of X with the multiplication being the concatenation of words.
  - ▹ A free monoid on a set X is a free semigroup on the disjoint union of X and a singleton called the identity, except that you cancel the indentity, i. e. factor out the equivalence relation generated by the identity axioms.
  - ▹ To get a free group on a set X you take the free semigroup on the disjoint union of X with itself and a singleton called the identity. Then you factor out the equivalence relation generated by all the axioms.
  - ▹ With abelian groups you throw in the commutativity axiom. In this case the counit takes a formal sum to a real sum in the particular group.
  - ▹ A free ring on a set X can be constructed by taking the free abelian group on the free monoid on X (by the distributive law any element can be expressed as a sum of products) and factoring out the equivalence relation generated by all the axioms.
  - ▹ If R is a ring, to get a free R-algebra on a set X you take a free abelian group on the free semigroup on the disjoint union of R and X etc.
  - ▹ A free ring with identity is a free $\mathbf{Z}$-algebra on the same set.
- □ Given a category $\mathcal{C}$ we can define a directed graph by taking the vertices to be the objects of $\mathcal{C}$ and directed edges the nontrivial morphisms. The left adjoint of this functor from CAT to DG is the free category functor. A free category on a directed graph Γ has as objects the vertices of Γ. As far as morphisms go, $[a, b]$ is the free monoid generated by the edges from a to b. The unit is the canonical inclusion of a directed graph into the directed graph of a free category generated by it. The counit is the identity on objects and takes a morphism in the free category on the directed graph of a category $\mathcal{C}$ (a formal composite of composable morphisms in $\mathcal{C}$) to the actual composition in $\mathcal{C}$.
- □ More examples:
  - ▹ Given $a \in \mathrm{Obj(SET)}$, we get an adjunction between $\mathrm{SET}[a, -]$ and $(a \times -)$. The counit $\epsilon_y\colon \mathrm{SET}[a, y] \to a$ is the evaluation map. The unit $\eta_y\colon y \to \mathrm{SET}[a, y \times a]$ takes a $\gamma \in y$ to the function $(\gamma, -)\colon a \to y \times a$.
  - ▹ In TOP we can do the same thing as above, if a is locally compact.

- ▷ The functor taking rings to multiplicative groups of units is the right adjoint for the group-ring functor.
- ▷ The inclusion functor from compact Hausdorff spaces into Tychonoff (completely regular T^1) spaces has a left adjoint called the Stone-Čech compactification. The unit is the canonical inclusion of a Tychonoff space in its compactification. What is the counit?
- ▷ Let CW_* be the category of connected pointed CW-complexes and base point preserving homotopy classes of continuous functions. The homotopy functor $\pi_1 : \mathrm{CW}_* \to \mathrm{GPS}$ is the left adjoint of the Eilenberg-MacLane functor. In particular π_1 preserves pushouts (Van Kampen's theorem).