Approximation of elliptic boundary value problems

History:

- 1950's: Finite differences and Rayleigh-Ritz-Galerkin
- FD: Young (1950) — over relaxation; faster iterative methods for large systems; 5-point schemes
- Courant: Variational method with piecewise linear basis functions leads to a 5-point scheme for the Laplace equation. (forgotten for 20 years)
- Decisive step: engineers independently develop finite elements (piecewise polynomial shape functions leads to FD)

Requirements for an appoximation:

- stability and optimal stability of approximate problems
- convergence of solutions, uniformity and optimal speed of convergence
- minimization of error
- sparsity and optimal condition number of matrices

Domain: $\Omega \subseteq \mathbf{R}^n$ — bounded open subset with smooth boundary Γ.

Differentiation: Let $p \in \mathbf{Z}_+^n$ with 1-norm. Define $D^p := \dfrac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}}$.
Space of test functions: $\mathscr{D}(\Omega) = \{u \in C^\infty(\Omega)$ with compact support in $\Omega\}$.
For a distribution f define $\dfrac{\partial f}{\partial x_i}$ by $\left\langle \dfrac{\partial f}{\partial x_i}, \varphi \right\rangle := \left\langle f, -\dfrac{\partial \varphi}{\partial x_i} \right\rangle \; \forall\, \varphi \in \mathscr{D}(\Omega)$

Differential operator: $\Lambda u := \sum\limits_{|p|,|q| \le k} (-1)^{|q|} D^q [a_{pq}(x) D^p u]$, where $a_{pq} \in L^\infty(\Omega)$,
Normal boundary derivatives $\gamma_j u$ (γ_0 is just restriction to Γ).

Sobolev space: $H^s(\mathbf{R}^n) := \left\{ u \in L^2(\mathbf{R}^n) \colon (1+|\eta|^2)^{\frac{s}{2}}\, \widehat{u}(\eta) \in L^2(\mathbf{R}^n) \right\} = \left\{ u \in L^2(\mathbf{R}^n) \colon D^p u \in L^2(\mathbf{R}^n), |p| \le s \right\}$
Let $H^s(\Omega)$ be the space of restrictions to Ω of functions in $H^s(\mathbf{R}^n)$. $H_0^s(\Omega) :=$ closure of $\mathscr{D}(\Omega)$ in $H^s(\Omega)$. $H^s(\Gamma) \cong H^s(\mathbf{R}^{n-1})$.

Trace theorem: $\gamma := (\gamma_0, \dots \gamma_{s-1}) \colon H^s(\Omega) \to \prod\limits_{j=0}^{s-1} H^{s-j-\frac{1}{2}}(\Gamma)$ is a bounded linear operator and $\ker \gamma = H_0^s(\Omega)$.

Energy product: a bilinear form $a(u,v) := \sum\limits_{|p|,|q| \le k} \int_\Omega a_{pq}(x)\, D^p u\, D^q v\, dx$

$H^k(\Omega, \Lambda) := \left\{ u \in H^k(\Omega) \colon \Lambda u \in L^2(\Omega) \right\}$

Green's formula: $\exists!$ linear operators $\delta_j \colon H^k(\Omega) \to H^{k-j-\frac{1}{2}}(\Gamma)$ $(k \le j \le 2k-1)$ such that $\forall\, u \in H^k(\Omega, \Lambda), v \in H^k(\Omega)$

$$a(u,v) = \int_\Omega \Lambda u \cdot v\, dx + \sum_{j=0}^{k-1} \int_\Gamma \delta_{2k-j-1}\, u\, \gamma_j v\, d\sigma(x)$$

Neumann problem: Given a forcing function $f \in L^2(\Omega)$ and boundary conditions $t_j \in H^{k-j-\frac{1}{2}}(\Gamma)$, $k \le j \le 2k-1$, we look for $u \in H^k(\Omega, \Lambda)$ with $\Lambda u + \lambda u = f$ and $\delta_j u = t_j$.

Equivalent formulation: Let $(u,v) := \int_\Omega u(x)\, v(x)\, dx$, $\langle f, g \rangle := \int_\Gamma f(x)\, g(x)\, d\sigma(x)$, $\ell(v) := (f,v) + \sum\limits_{j=0}^{k-1} \langle t_{2k-j-1}, \gamma_j v \rangle$.
u is a solution of the Neumann problem $\Leftrightarrow u \in H^k(\Omega)$ and $a(u,v) + \lambda(u,v) = \ell(v)\; \forall v \in H^k(\Omega)$

General problem: Suppose $V \subseteq H$ are Hilbert spaces, V is compact and dense in H.
Let a and ℓ be continuous bilinear and linear forms on V. Find $u \in V$ such that $a(u,v) + \lambda(u,v) = \ell(v)\; \forall v \in V$.

Existence-uniqueness theorem: Suppose a is V-elliptic, i.e. $a(v,v) \ge c\, |v|_V^2\;\; \forall v \in V$ and some constant c. If λ is not in the spectrum of a (a countable set of isolated points), then the solution exists and is unique.

Proof: The result follows from the Lax-Milgram theorem and the Riesz-Fredholm theorem.

References:

J-P. Aubin, *Approximation of elliptic boundary-value problems*, Wiley-Interscience, 1972
J.F. Botha, G.F. Pinder, *Fundamental concepts in the numerical solution of differential equations*, Wiley, 1983
A.H. Zemanian, *Distribution theory and transform analysis*, McGraw-Hill, 1965 (Dover, 1987)