Eigenvalues and eigenspaces

Given a vector space V, a subspace W, a linear operator $\mathscr{L}: W \to V$, and a constant λ, we say that λ is an *eigenvalue* of $\mathscr{L}$ (relative to W) if the operator $\lambda I - \mathscr{L}: W \to V$ has a nontrivial kernel (null space). The kernel is called the *eigenspace* corresponding to λ and its nontrivial elements are called *eigenvectors* (or *eigenfunctions*).

Example 1: Let $V = W = \mathscr{C}^\infty[0, L]$ and let $\mathscr{L}u = u_x$. For any λ the eigenspace has dimension 1 and is generated by $e^{\lambda x}$.

Example 2: Let $\mathscr{D}u = \mathscr{L}(\mathscr{L}u) = u_{xx}$. If $\lambda \neq 0$, the eigenspace has dimension 2 and is generated by $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$. If $\lambda = 0$, the eigenspace is generated by 1 and x.

Example 3: Let $W = \{u : u(0) = u(L) = 0\}$, then $\lambda = -\omega^2$, where $\omega = \dfrac{n\pi}{L}$, $n \in \mathbf{Z}$, with eigenspace generated by $\sin(\omega x)$.

Adjoints of operators

If V has an inner (dot) product, then given a linear operator $\mathscr{L}: W \to V$, the *adjoint* operator $\mathscr{L}^*: W \to V$ is defined by $(\mathscr{L}u) \cdot v = u \cdot (\mathscr{L}^* v)$ for all u and v in W. If $\mathscr{L}^* = \mathscr{L}$, then $\mathscr{L}$ is called *self-adjoint.*

Example 1: Let $V = W = \mathbf{C}^n$ with $u \cdot v = u^t \overline{v}$. Given a matrix A, $(Au) \cdot v = (Au)^t \overline{v} = u^t A^t \overline{v} = u \cdot (\overline{A}^t v)$, so $A^* = \overline{A}^t$. Self-adjoint matrices are called Hermitian. [1]

Example 2: Let $V = \mathscr{C}^\infty[0, L]$, $\mathscr{L}u = u_x$, $\mathscr{D}u = u_{xx}$, and $W = \{u\colon u(0) = u(L) = 0\}$.
Then $(\mathscr{L}u) \cdot v = \displaystyle\int_0^L u_x \overline{v}\, dx = u\overline{v}\Big|_0^L - \int_0^L u\overline{v}_x\, dx = -\int_0^L u\overline{v}_x\, dx = -u \cdot (v_x)$, so $\mathscr{L}^* = -\mathscr{L}$. Similarly, $\mathscr{D}^* = \mathscr{D}$.

Example 3: Let Ω be a domain in $\mathbf{R}^n$, $V = \mathscr{C}^\infty\Omega$, $W = \left\{u\colon\ u\Big|_{\partial\Omega} = 0\right\}$. Then the Laplacian operator on W is self-adjoint.

Proof: (real-valued case) By Green's second identity [2] $\displaystyle\int_\Omega (u\nabla^2 v - v\nabla^2 u)\, dV = \int_{\partial\Omega} (u\nabla v - v\nabla u) \cdot dA = 0.$

Theorem 1: Eigenvalues of a self-adjoint operator $\mathscr{L}$ are real.

Proof: Let λ be an eigenvalue of $\mathscr{L}$ and u an eigenvector. Then $(\mathscr{L}u) \cdot u = (\lambda u) \cdot u = \lambda(u \cdot u)$.
On the other hand $u \cdot (\mathscr{L}u) = u \cdot (\lambda u) = \overline{\lambda}(u \cdot u)$. Thus, $\lambda = \overline{\lambda}$.

Theorem 2: Eigenvectors of a self-adjoint operator $\mathscr{L}$ corresponding to distinct eigenvalues are orthogonal.

Proof: Let eigenvector u correspond to eigenvalue λ and v to μ. Then $(\mathscr{L}u) \cdot v = (\lambda u) \cdot v = \lambda(u \cdot v)$.
On the other hand $u \cdot (\mathscr{L}v) = u \cdot (\mu v) = \overline{\mu}(u \cdot v) = \mu(u \cdot v)$. Since, $\lambda \neq \mu$, $u \cdot v = 0$.

[1] Charles Hermite (1822–1901)
[2] George Green (1793–1841)