PLIMPTON 322: A REMARKABLE ANCIENT BABYLONIAN TABLET ON NUMBER THEORY

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The oldest surviving document on number theory.

Part of a baked clay tablet made in Babylon between 1900 & 1600 BC.

In the G.A. Plimpton Collection at Columbia University, New York.

Plimpton bought it from Banks in 1923.

Thought to have been found at Senkereh.

Deciphered by Neugebauer & Sachs in 1945.
<u>Math. Cuneiform Texts</u>, Yale Univ. Press (1945) 29-35.

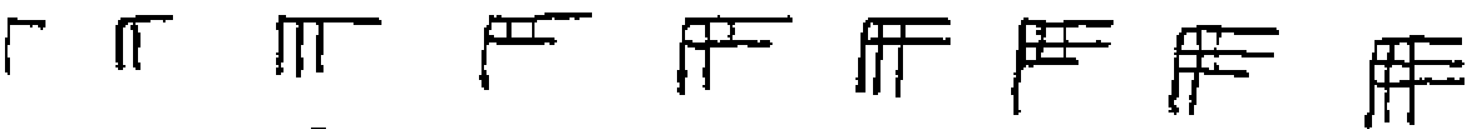
Traces of modern glue on left edge, but alas the left fragment is now lost.

Survey paper by J. Friberg in 1981
Methods & traditions in Babylonian Maths,
<u>Historia Mathematica</u> 8 (1981) 277-318

CUNEIFORM SCRIPT

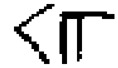
Γ means 1 } pressed with a stylus into soft clay.
< means 10 }

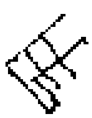
space means 0



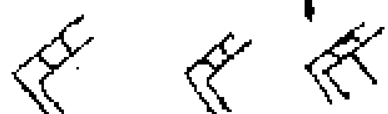
1 2 3 4 5 6 7 8 9

10 20 30 40 50

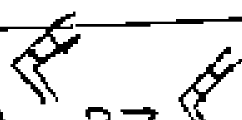
Decimal notation (up to 59 only) 12 = 



37 $\stackrel{?}{=}$ 

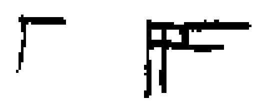


SEXAGESIMAL NOTATION



Integers

$97 = (1\times 60) + 37 \stackrel{?}{=} 1,37 =$ 

$65 = (1\times 60) + 5 = 1,5 =$ 

$12709 = (3\times 60^2) + (31\times 60) + 49$

$= 3,31,49 =$ 

Sexagesimal $\frac{9}{16} = \frac{33}{60} + \frac{45}{60^2} = 33,45 =$ 

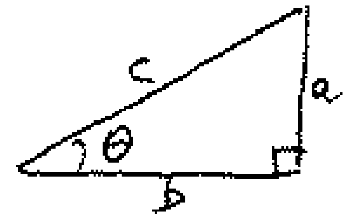
Question: What is the meaning of the tablet?

Answer: It's a list of Pythagorean triples.

Pythagoras' Theorem

In a right-angle triangle

$$a^2 + b^2 = c^2$$



Definition A *Pythagorean triple* (a, b, c) is a coprime triple of positive integers such that

$$a^2 + b^2 = c^2$$

Here *coprime* means they have no common factors.

Examples $(3, 4, 5)$

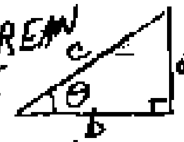
$(65, 72, 97)$ — line 5

$(12709, 13500, 18541)$ — line 4

$12709 = 3, 31, 49$ $\quad$ $18541 = 5, 9, 1$

Conclusions ① Columns 2 & 3 are two sides of a Pythagorean triple.

② The third side is "regular" in the sense that it only has prime factors 2, 3, 5

TABLET: SEXAGESIMAL NOTATION				PYTHAGOREAN TRIPLE 			ANGLE	COPRIME PAIR	
	shortest side a	diagonal c	Line	a	b	c	θ°	x	y
59, 0, 15	1, 59	2, 49	1	119	120	169	44.8	5	12
56, 56, 58, 14, 50, 6, 15	56, 7	1, 20, 25	2	3367	3456	4825	44.3	27	64
55, 7, 41, 15, 33, 45	1, 16, 41	1, 50, 49	3	4601	4800	6649	43.8	32	75
53, 10, 29, 32, 52, 16	3, 31, 49	5, 9, 1	4	12709	13500	18541	43.3	54	125
48, 54, 1, 40	1, 5	1, 37	5	65	72	97	42.1	4	9
47, 6, 41, 40	5, 19	8, 1	6	319	360	481	41.5	9	20
43, 11, 56, 28, 26, 40	38, 11	59, 1	7	2291	2700	3541	40.3	25	54
41, 33, 45, 14, 3, 45	13, 19	20, 49	8	799	960	1249	39.8	15	32
38, 33, 36, 36	8, 1	12, 49	9	481	600	769	38.7	12	25
35, 10, 2, 28, 27, 24, 26, 40	1, 22, 41	2, 16, 1	10	4961	6480	8161	37.4	40	81
33, 45	3	5	11	3	4	5	36.9	1	2
29, 21, 54, 2, 15	27, 59	48, 49	12	1679	2400	2929	35.0	25	48
27, 0, 3, 45	2, 41	4, 49	13	161	240	289	33.9	8	15
25, 48, 51, 35, 6, 40	29, 31	53, 49	14	1771	2700	3229	33.3	27	50
23, 13, 46, 40	28	53	15	28	45	53	31.9	5	9
22, 7, 19, [illegible]	2, 55	5, 37	16	175	288	337	31.3	9	16

Questions
1. What does the first column mean?
2. How did the author discover all these triples?
3. Why did she choose just these?
4. Why did she put them in this order?

DIGRESSION

Fermat's Last Theorem

$a^n + b^n = c^n$ has no integer solutions for $n \geq 3$.

Fermat (1601 – 1665) stated this result as a marginal note in his copy of Diophantus around 1637. What he actually wrote was (in Latin):

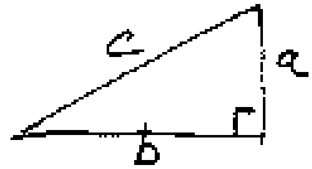
"To resolve a cube into the sum of two cubes, a fourth power into fourth powers, or in general any power higher than the second into two of the same kind, is impossible; of which fact I have found a remarkable proof. The margin is too small to contain it."

He subsequently published the proof for $n = 4$.

It is called his "last" theorem, because it is the last of the many results that he stated without proof to be subsequently proved & published by someone else.

Andrew Wiles announced a proof on 23 June 1993 at the Isaac Newton Institute in Cambridge

Let (a,b,c) be a Pythagorean triple



<u>Lemma 1</u> a, b, c are pairwise coprime.

<u>Proof</u> If any two have a common factor then it must also be a factor of the third, by the equation

$$a^2 + b^2 = c^2$$

<u>Remark</u> Not true for <u>non</u>-Pythagorean triples, eg $(4,5,6)$

<u>Lemma 2.</u>
c is odd, and one of a, b is even and the other odd.

<u>Proof</u> At most one of a, b, c is even by Lemma 1.

At least one of them is even, otherwise all three are odd, and then a^2+b^2 is even while c^2 is odd, a contradiction.

Therefore exactly one is even.

If c is even then a, b are odd, so we can write

$$\begin{aligned} a &= 2\alpha+1 & \therefore\ a^2 &= 4\alpha^2+4\alpha+1 \\ b &= 2\beta+1 & b^2 &= 4\beta^2+4\beta+1 \\ c &= 2\gamma. & c^2 &= 4\gamma^2. \end{aligned}$$

Therefore $a^2+b^2 = 4(\alpha^2+\alpha+\beta^2+\beta)+2$, <u>not</u> divisible by 4 whereas c^2 is divisible by 4, a contradiction

Therefore c is odd.

Therefore one of a, b is even & the other odd.

Remark If a is even we can interchange a, b and make b even. Therefore assume for the moment that b is always even.

Definition

Let T = the set of Pythagorean triples with b even.

Definition

Let P = the set of coprime pairs of positive integers (x,y), such that $x<y$ and one of them is even.

Theorem 1. There is a one-to-one map $f: P \longrightarrow T$ given by $(x,y) \longmapsto (a,b,c)$ where $\begin{cases} a = y^2 - x^2 \\ b = 2xy \\ c = x^2+y^2 \end{cases}$

Example $(1,2) \longrightarrow (3,4,5)$

Remark This gives us a systematic way of listing all Pythagorean triples.

Proof of Theorem 1

① We first check that f does map P into T.
In other words, given $(x,y) \in P$ and

$$\text{given} \begin{cases} a = y^2 - x^2 \\ b = 2xy \\ c = x^2 + y^2 \end{cases}$$

We have to verify that (a,b,c) satisfies Pythagoras' equation, is coprime, & b is even.

$$a^2 + b^2 = (y^2 - x^2)^2 + 4x^2y^2 = (x^2+y^2)^2 = c^2,$$

as required.

Since one of x, y is even, both a, c are odd.
Therefore if p is a common factor of a, c it must be odd.

$\therefore$ p is also a factor of $\begin{cases} c - a = 2x^2 \\ c + a = 2y^2 \end{cases}$

$\therefore$ p is a common factor of x, y.

But x, y are coprime. $\therefore$ a, c cannot have a common factor.
$\therefore$ a, c coprime $\therefore$ (a,b,c) - coprime

Finally $b = 2xy$ & so b is even.

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② Next we verify that f is injective.
i.e. that distinct pairs map to distinct triples.
Suppose not. Suppose that distinct pairs (x,y) & (x',y') map to the same triple. Then

$$x = \sqrt{\frac{c-a}{2}} = x', \quad y = \sqrt{\frac{c+a}{2}} = y'$$

Therefore they are the same after all. $\therefore$ f injective.

(3) Finally we must show that f is surjective.

In other words, given a triple (a,b,c) we have to find a pair (x,y) that maps onto it.

First let $X = \frac{c-a}{2}$, $Y = \frac{c+a}{2}$. Then $0 < X < Y$.

Note that X, Y are integers because both a, c are odd.

Note also that $X+Y = c$, $Y-X = a$.

Therefore X, Y are coprime, for if they had a common factor so would a, c, contradicting Lemma 1.

Let $Z = \frac{b}{2}$, which is an integer since b is even.

Then $XY = \frac{c-a}{2} \cdot \frac{c+a}{2} = \frac{c^2-a^2}{4} = \frac{b^2}{4} = Z^2$.

If p is a prime factor of X, then p divides Z^2.

$\therefore$ p divides Z. $\therefore$ an even power of p divides Z^2.

But p does not divide Y, since X, Y are coprime.

$\therefore$ an even power of p divides X.

This is true for all prime factors of X.

$\therefore X = x^2$, for some integer x.

Similarly $Y = y^2$, for some integer y.

Now x, y are coprime because X, Y are, and $x < y$ because $X < Y$.

By construction
$$\begin{cases} a = Y - X = y^2 - x^2 \\ b = 2Z = 2\sqrt{XY} = 2xy \\ c = X + Y = x^2 + y^2 \end{cases}$$

Finally x, y are not both odd, otherwise a, c would be even.

Hence we have found a pair $(x,y) \in P$ that maps onto the given triple $(a,b,c) \in T$.

Hence f is surjective, as required.

This completes the proof of Theorem 1.