A brief history of topology

by E.C. Zeeman

Berkeley, 23 October 1993

On the occasion of Max Kneser's 60th birthday
HISTORY OF TOPOLOGY

1890 - applied origins
1900 - analytic
1910 - geometric
1920 - geometric
1930 - algebraic
1940 - algebraic
1950 - analytic
1960 - geometric
1970 - differential
1980 - applications
1990 - ?
2000 - ?

- 1895 Poincare: Analysis Situs
- 1930 Lefschetz: Topology
- 1942 " : Algebraic Topology
- ENTER MOE
- 1960 geometric resurgences
dynamical systems
1924

(1) Emil Artin: Knots $S^2 \subset \mathbb{R}^4$

(2) Alexander: You can't knot a torus on both the outside & the inside at once.

(3) Alexander: horned sphere
3 MAJOR PROBLEMS

1. **MANIFOLD PROBLEM**: classify $n$-manifolds
   BLOCKED by the Poincaré Conjecture 1899
   UNBLOCKED by Smale in 1961 in dimensions $\geq 5$

2. **EMBEDDING PROBLEM**: Given $M$, what
   in the least dimension $q$, $M \subset \mathbb{R}^q$?

3. **KNOTTING PROBLEM**: Given $M$, $Q$
   classify embeddings, $M \subset Q$
   up to isotopy, $[e.g., S^1 \subset \mathbb{R}^3 \& \text{(6)}]$
   BLOCKED by the Alexander Horned Sphere 1924
   UNBLOCKED by Barry Mazur/Mont Brown 1959
1931 Heising Hopf.

Is the 4-cell in $\mathbb{C}P^2$ attached essentially?

Real projective plane

$\mathbb{R}P^2 = e_0 \cup e_1 \cup e_2$

$\mathbb{R}P^1 = S^1$

$\dim 0 \quad 1 \quad 2$

Homology $\mathbb{Z} \quad \mathbb{Z}_2 \quad 0$

Attaching map, degree 2, ... essential

Complex projective plane

$\mathbb{C}P^2 = e_0 \cup e_2 \cup e_4$

$\mathbb{C}P^1 = S^2$

$\dim 0 \quad 1 \quad 2 \quad 3 \quad 4$

Homology $\mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}$

Chern character $\mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}$

(1936) $x \quad x^2 = y$

Theorem $H_3(S^2) = \mathbb{Z}$

$e_2 \quad e_0 \quad e_4$

Attaching map $S^3 \to S^2$

Linking number invariant

$S^3 \to S^2$
ALGEBRAIC TOPOLOGY

1740's Euler numbers
1871 Betti numbers
1920's Emmy Noether: homology groups
1930's Čech: topological invariance
1936 Alexander: cohomology ring
1940 homotopy groups
1951 Serre's thesis
1953 axiomatized steenrod
1950's cohomology groups of groups

MacLane Eisenberg

geometric topology
**ALEXANDER COMPACT COHOMOLOGY**

**Definition**

\[ A^n = \text{all maps } f : X^{n+1} \to \mathbb{Z} \]

\[ \text{Supp } f = \bigcup \{ x \in X / f \text{ nil } N, f|N^{n+1} = 0 \} \]

\[ C^n = \{ f \text{ with compact support} \} / \{ f \text{ with empty support} \} \]

\[ H(X) = H(C, \mathcal{E}) \]

**Properties**

Some useful facts:

- Exact
- Perfect excision: \( H(X, Y) = H(X - Y) \)
- If \( Y \) closed in \( X \).
- Connectable (on countable complexes)

\[ H^2(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & q = n \\ 0, & q \neq n \end{cases} \]

**Advantages**

- Countable bouquet of 2-spheres
- Stable/diffeo repellor
- Morse inequalities

- Attractor
- Instable saddle
- Attractor
Embedding Problem

Bundles
Tangent Bundles
Characteristic classes Stiefel-Whitney
Obstructions

\[ M^n \subset \mathbb{R}^2 \Rightarrow T \oplus N = 1 \]
embedding \[ \Rightarrow g \geq g_0 \]
\[ M^n \subset \mathbb{R}^2 \Rightarrow g \geq g_0 \]

immersion
Therefore it's really an immersion problem.
The creation of a draft subject.
The tail wags the dog.

ENTER MOE.
GEOMETRIC RESURGENCE

1956 Milnor: Diff structures on $S^7$

1959 Mauzer/Brown: locally flat $S^n \subset \mathbb{R}^{n+1}$ unknotted.

1960 Zeeman: $S^2 \subset \mathbb{R}^5$ unknotted.

1961 Smale: Poincaré Conjecture $\dim \geq 5.$
Theorem \( S^2 \subset R^5 \) unknotted.

Proof: Look at \( S^2 \) from \( V \) in good position.
If only finitely many \( \{ V N_i F_i \}, i = 1, \ldots, n \) new fan.

Let \( A = \text{disk } \# N_i, \# F_i \)
\( B = \text{complementary disk} \)
Then \( S^2 = A \cup B \)
\( \cong V(\partial A) \cup B \)
\( = \chi(VB) \)

1961 Theorem (Zeeman) \( S^3 \subset R^6 \) unknotted
1961 Theorem (Haefliger) \( S^3 \subset R^6 \) knotted