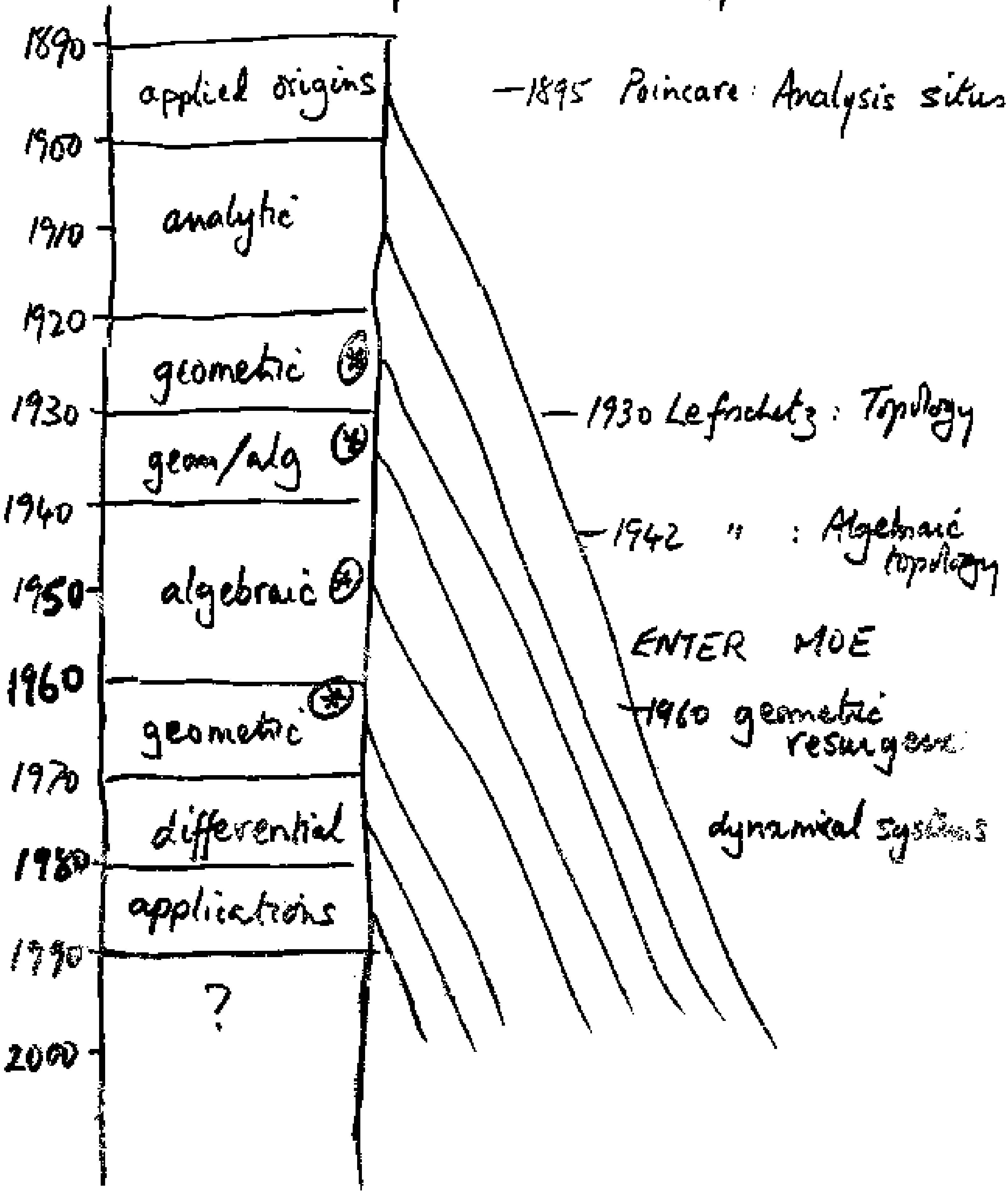
A brief history of topology

by E.C. Zeeman

Berkeley, 23 October 1993

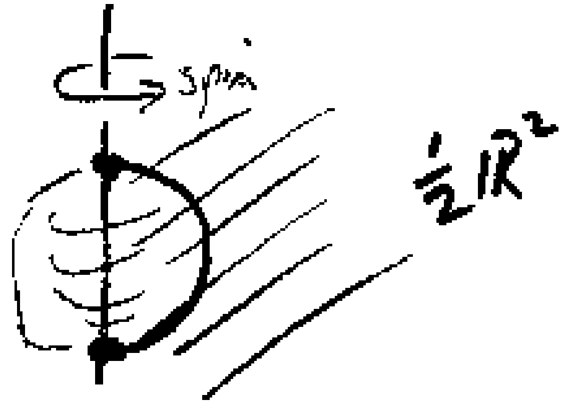
On the occasion of Moe Hirsch's 60th birthday

HISTORY OF TOPOLOGY

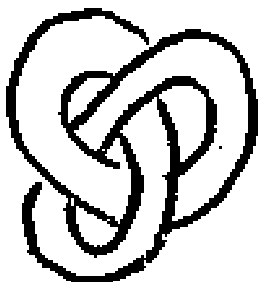


$\boxed{1924}$

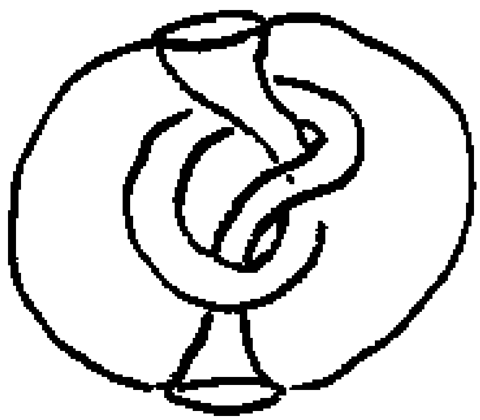
(1) Emil Artin : Knots $S^2 \subset \mathbb{R}^4$



(2) Alexander : You can't knot a torus on both the outside & the inside at once

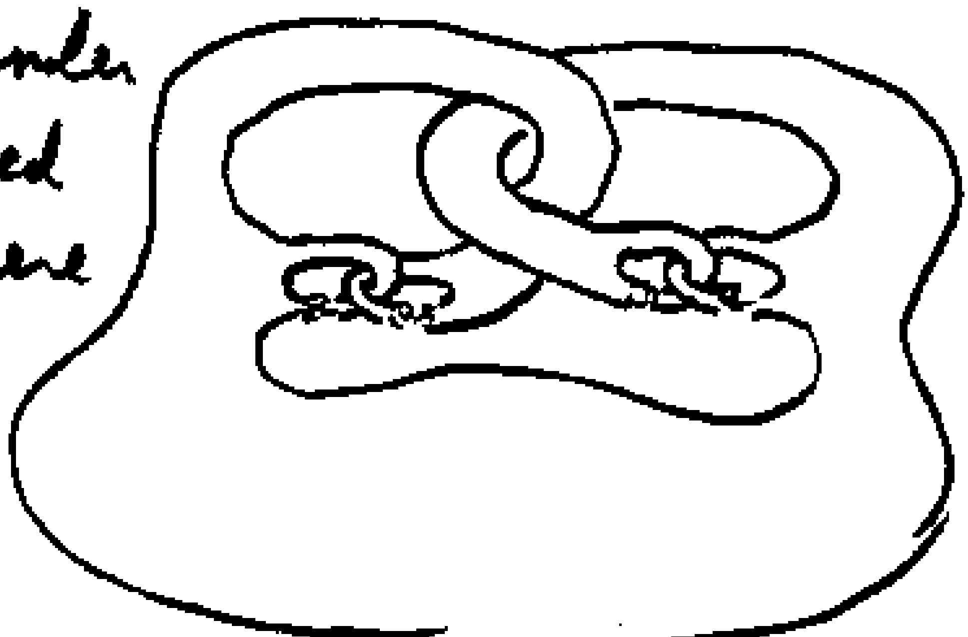


knotted on the outside



knotted on the inside.

(3) Alexander horned sphere



3 MAJOR PROBLEMS in 20th

① **MANIFOLD PROBLEM**: classify n-manifolds

BLOCKED by the Poincaré Conjecture 1899

UNBLOCKED by Smale in 1961 in dimensions ≥ 5

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② **EMBEDDING PROBLEM**: Given M, what is the least dimension q, $M \subset \mathbb{R}^q$?

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③ **KNOTTING PROBLEM**: Given M, Q

classify embeddings $M \subset Q$

up to isotopy. [e.g. $S^1 \subset \mathbb{R}^3$  ....]

BLOCKED by the Alexander Horned Sphere 1924

UNBLOCKED by Barry Mazur/Mort Brown 1959

$\boxed{1931}$ Heinz Hopf.

Is the 4-cell in $\mathbb{CP}^2$ attached essentially?

Real projective plane

$$P^2 = \underbrace{e_0 \cup e_1}_{P^1 = S^1} \cup e_2$$

dim	0	1	2
homology	$\mathbb{Z}$	$\mathbb{Z}_2$	0

$$\partial e_2 = S^1 \longrightarrow S^1$$

attaching map, degree 2, ∴ essential

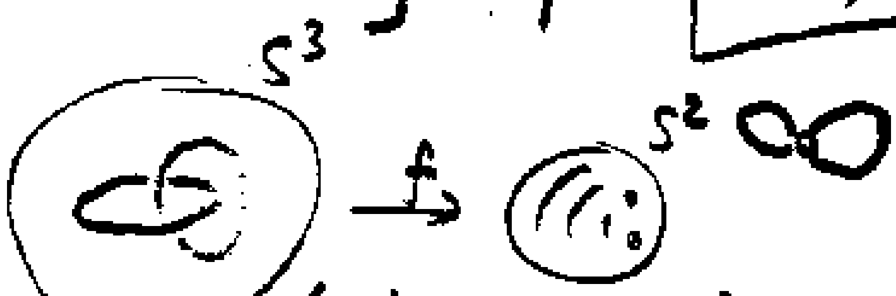
Complex projective plane

$$\mathbb{CP}^2 = \underbrace{e_0 \cup e_2}_{\mathbb{CP}^1 = S^2} \cup e_4$$

dim	0	1	2	3	4
homology	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
cohomology (1936)	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
			x		$x^2 = y$

$$\partial e_4 = S^3 \xrightarrow{f} S^2$$

attaching map



Linking # invariant

Theorem

$$\pi_3(S^2) \cong \mathbb{Z}$$

ALGEBRAIC TOPOLOGY

1760's Euler number

1871 Betti numbers

1920's Emmy Noether: homology groups

1930's Cech: topological invariance

1936 Alexander: cohomology ring

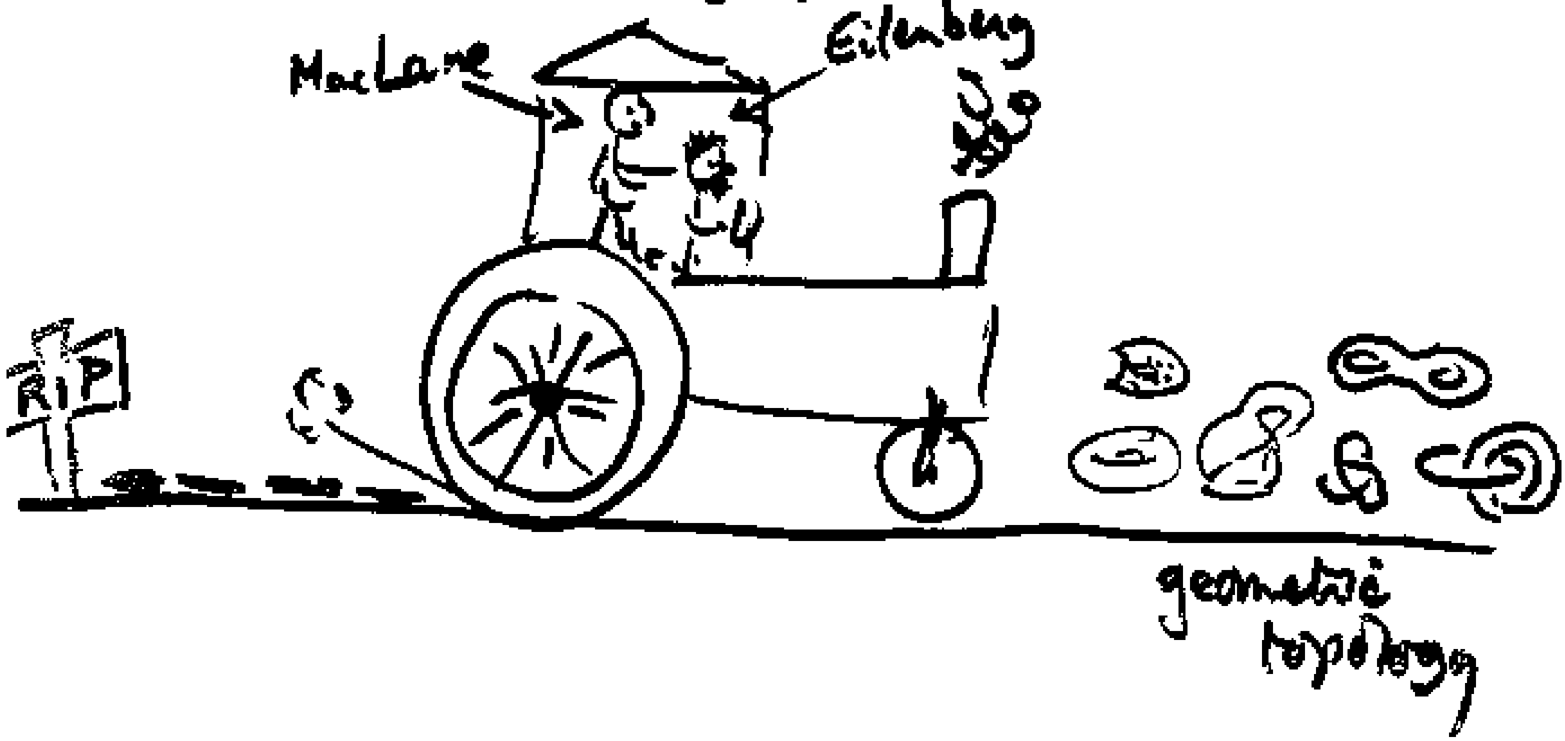
1940 homotopy groups

1951 Serre's thesis

1953 axiomatic steamroller

1950's cohomology groups of groups

dominance of singular homology



ALEXANDER COMPACT COHOMOLOGY

Definition A^n = all maps $f : X^{n+1} \longrightarrow \mathbb{Z}$

supp $f = \{ x \in X \mid \nexists$ nbd N, $f|N^{n+1} = 0 \}$

$C^n = \{f$ with compact support$\} / \{f$ with empty support$\}$

$H(X) = H(C, \delta)$

Properties Same as Čech

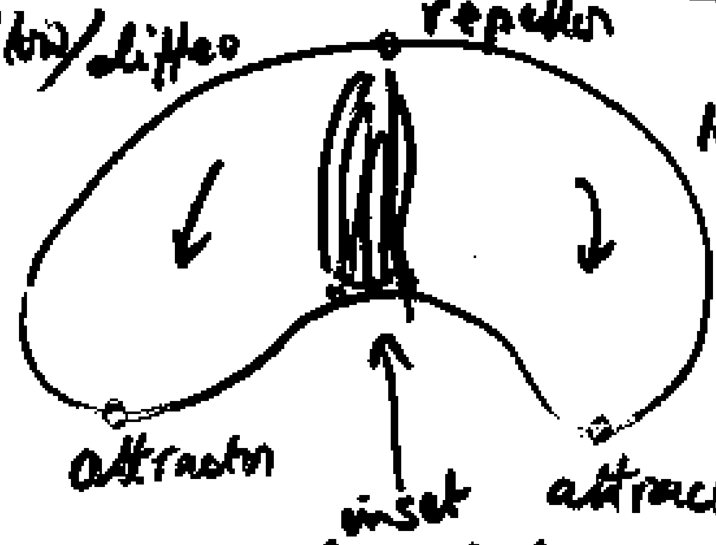
- Exact
- Perfect excision: $H(X, Y) = H(X - Y)$ if Y closed in X.
- Countable (on countable complexes)

$$H^q(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & q = n \\ 0, & q \neq n \end{cases}$$

Advantages

Countable bouquet of 2-spheres

flow/diffeo



Morse inequalities

1940's & 50's EMBEDDING PROBLEM

Bundles

Tangent Bundles

Characteristic classes Stiefel-Whitney
Chern

Obstructions

$M^n \subset \mathbb{R}^q \Rightarrow T \oplus N = 1$

embedding $\Rightarrow q \geq q_0$

$M^n \propto \mathbb{R}^q \Rightarrow q \geq q_0$

immersion

Therefore its really an immersion problem

The creation of a daft subject.

The tail wags the dog.

ENTER MOE.

GEOMETRIC RESURGENCE

1956 Milnor : Diff structures on S^7

1959 Mazur/Brown : locally flat $S^n \subset \mathbb{R}^{n+1}$ unknotted.

1960 Zeeman : $S^2 \subset \mathbb{R}^5$ unknotted

1961 Smale : Poincaré C; dim ≥ 5.

Theorem $S^2 \subset \mathbb{R}^5$ unknotted.

Proof Look at S^2 from V in gen position:

$\exists$ only finite # crossings VN_iF_i , $i=1,\dots,n$

near far.

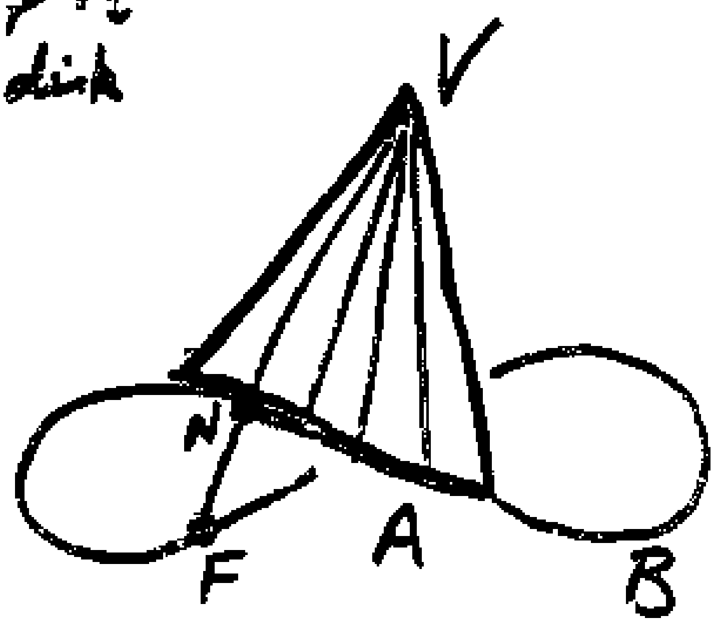
Let A = disk $\ni N_i$, $\not\ni F_i$

B = complementary disk

Then $S^2 = A \cup B$

$\overset{VA}{\simeq} V(\partial A) \cup B$

$= \partial(VB)$



1961 Theorem (Zeeman) $S^3 \subset \mathbb{R}^6$ unknotted

1961 Theorem (Haefliger) $S^3 \subset \mathbb{R}^6$ knotted !