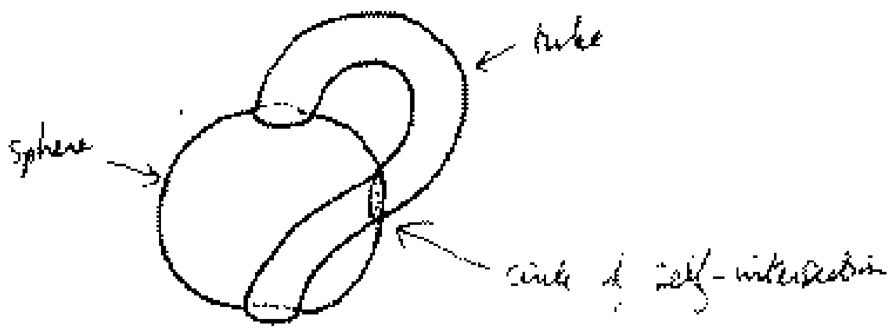
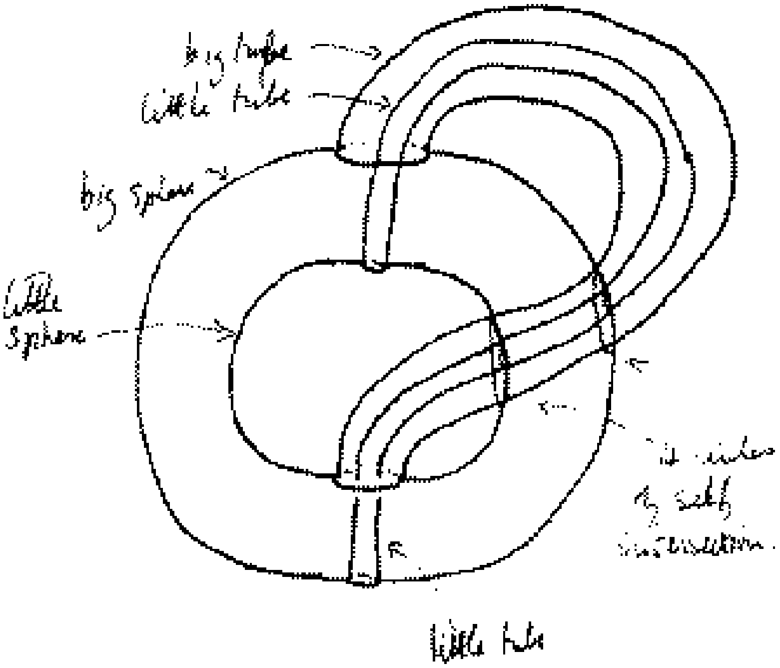
Derek Hacon Turning a sphere inside out. In other words a regular homotopy (= 1-parameter family of immersions) of S^2 in $\mathbb{R}^3$, that brings it back onto itself with degree -1. We start by turning a torus inside out.

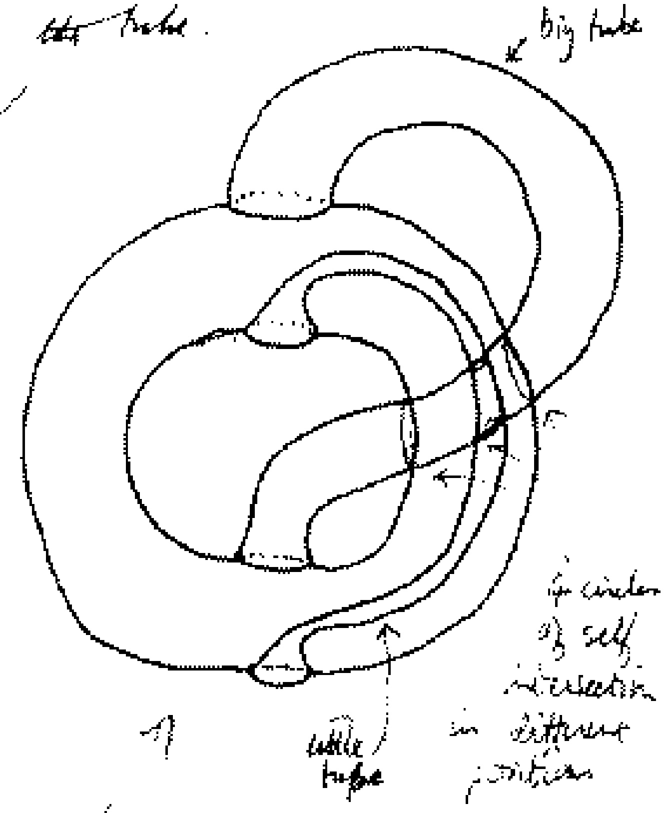
Step ① Consider a Klein bottle, immersed with 1 circle of self intersection.



Consider the double-cover of the Klein bottle, which is a torus wrapped twice round the Klein bottle, & which can be turned inside out without moving it.

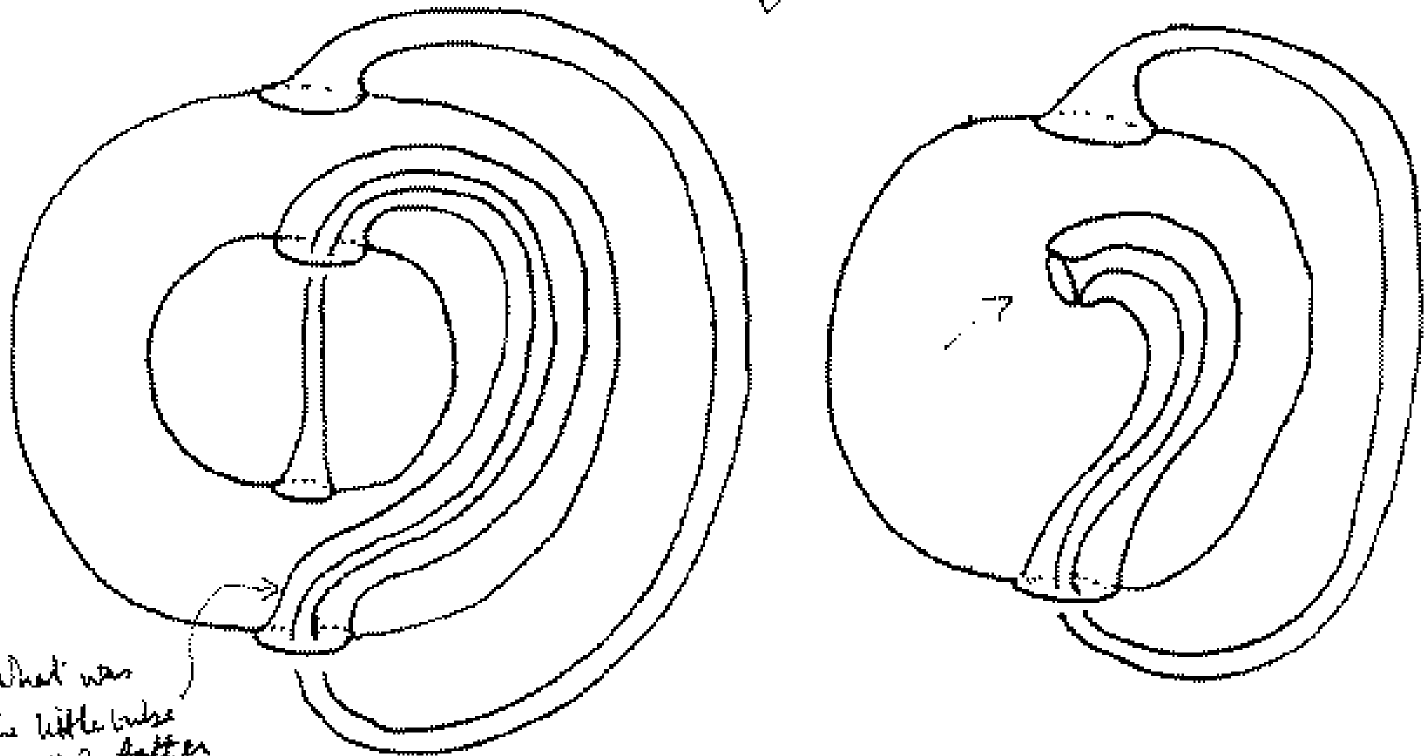
Step ② Perturb the immersion of ~~the~~ torus as shown by doubling the sphere & doubling ~~the~~ tube.



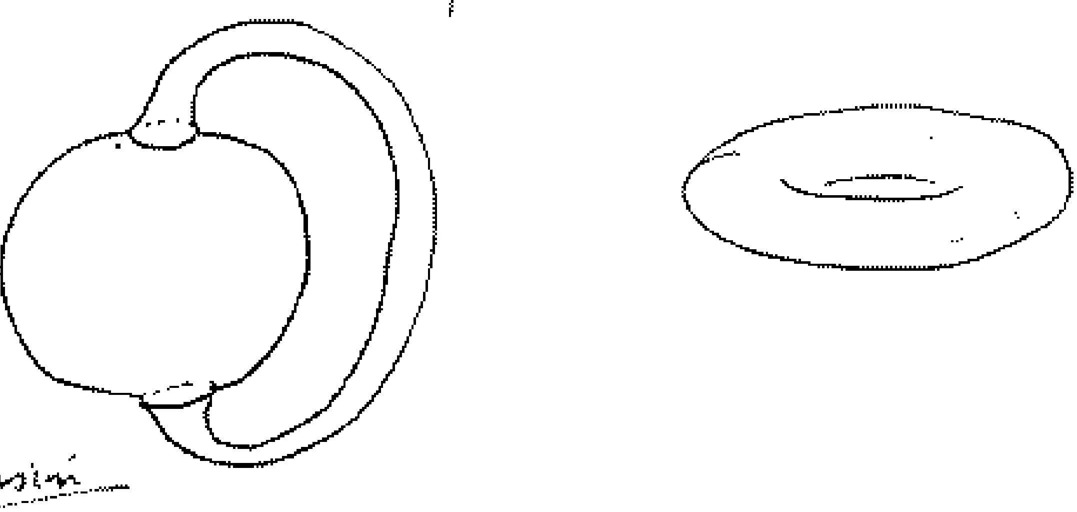


Step ③ Isotop the little tube keeping everything else fixed.

Step ④ Now isotop the big tube until it runs along inside the little tube, while keeping everything else fixed, giving an embedding of the torus.



Step ⑤ Retract by an isotopy (= 1-parameter family of embeddings) the little sphere & two tubes inside the big sphere, giving an ordinary torus.



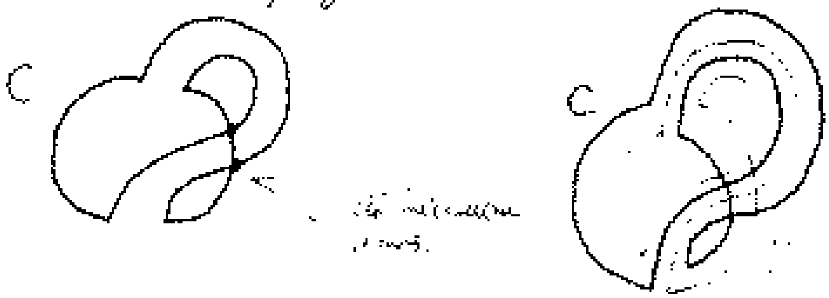
Conclusion

Steps $5 \to 4 \to 3 \to 2 \to 1 \to 1 \to 2 \to 3 \to 4 \to 5$

turn the torus inside out.

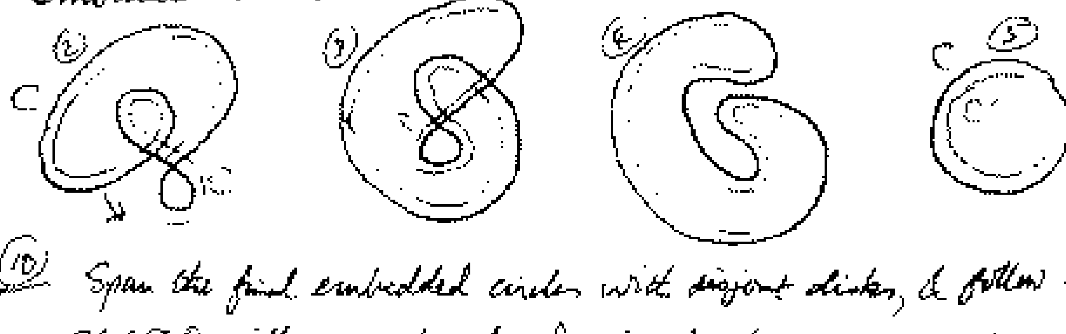
Step ⑥ The paper is a plane of symmetry of all pictures and regular homotopies. Therefore cut the pictures in half by this plane of symmetry, discard what lies in front of the plane, & retain only what lies behind the plane. In Step ① we have retained half the Klein bottle, which is a Möbius strip, M

Step ⑦ The boundary of M is an immersed circle C in the plane.



Step ⑧ In step ② C becomes two immersed circles C and C′ bounding half the torus behind the plane, which is an immersed cylinder.

Step ⑨ In steps ③④⑤ the two circles regularly homotope into embedded circles.



Step ⑩ Span the final embedded circles with disjoint disks, & follow steps ⑤④③②① with regular homotopies of these disks onto a single immersed disk spanning $C = \partial M$ in Step ①.

Then we have step ⑤ = cylinder ∪ 2 disks = embedded sphere

step ① = Möbius strip ∪ 1 disk = immersed real projective plane.

Therefore ⑤ ⟶ ① ⟶ ① ⟶ ⑤ gives the required inversion of the sphere.

turn inside out without moving