A Dynamics for Power and Control in Society

We adapt the macro structural approach to a society. There are no individuals.

The structures

King/Dictator
Government
People/Peasants
People/Middle Class , etc
Police/Army
Education
Church
Economy/Industry
Labor.
⋮

Three axioms of societal power and control.

1. Each of the structures of a society exerts or attempts to exert power to control the others

2. Each of the structures accedes to the others a certain fraction of what would be complete self control.

3. Each structure acts in a way to align the fraction of control that another structure has over a third structure according to an intrinsic control matrix. This matrix varies from structure to structure.

These axioms are independent of particular political theories.

Only the coefficients and structure set vary from political system to political system.

Goals: To develop a model that is consistent with accepted political observations vis á vis

A. Revolutions.

B. Emergent structures.

C. Anarchy.

D. Mediation.

E. Passivity.

F. Conformability.

G. Extinction.

Each of these "events" should be interpretted through the model. In turn, the analysis should generate reasonable conclusions.

The Naive Model.

Denote the structures by $S_1, S_2, \cdots, S_n$.

Let x_i be the fraction of control that structure S_i has over some facet of society,

So

$$x_1 + x_2 + \cdots + x_n = 1.$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

p_{ij} - that fraction of control acceded by S_j to S_i.

$$\sum_{i=1}^{n} p_{ij} = 1 \qquad j = 1, 2, \cdots, n$$

r_j - the power, or resources, possessed by S_j to accomplish its desiderata.

Naive Model

$$\dot{x}_i = \sum_{j=1}^{n} r_j (p_{ij} - x_i) x_j, \quad i = 1, 2, \cdots, n.$$

$R = \mathrm{diag}(r_1, r_2, \cdots, r_n)$

$r = [r_1 \cdots r_n]^T$

$P = p_{ij}, \quad 1 \le i, j \le n$

$$\boxed{\dot{x} = (PR - r \otimes x)x}$$

Naive Model

Similar to: prey-predator systems

Combined cooperative-competitive

Note $\dot{x}_i = \sum_{j=1}^{n} r_j (p_{ij} - x_i) x_j \qquad i = 1, \cdots, n$

Sum in i

$$\sum_i \dot{x}_i = \sum_{i=1}^{n} \sum_{j=1}^{n} r_j (p_{ij} - x_i) x_j$$

$$= \sum_{j=1}^{n} r_j x_j \sum_{i=1}^{n} (p_{ij} - x_i)$$

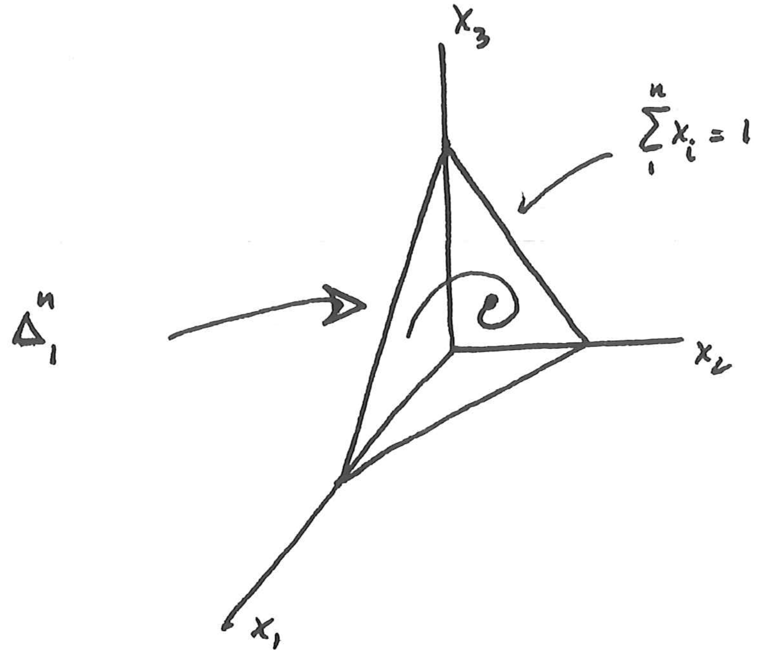
$$= \sum_{j=1}^{n} r_j x_j \cdot 0$$

$$= 0$$

Hence if $x(0)$ satisfies $\sum_{i=1}^{n} x_i(0) = 1$, it follows that

$$\sum_{i=1}^{n} x_i(t) = 1 ,$$

for all t.



<u>Basic Goals</u>: ① To determine equilibria.

② To establish some sort of stability.

---

③ To analyse special circumstances of the data.

Why is $\dot{x} = (rR - r \otimes x)x$ called the naive model?

General Model.

Begin with three structures S_i, S_j and S_k.

Control axiom. S_k accedes to S_j a certain control fraction over S_i defined by

$$p_{ijk}$$

To allocate all control:

$$\sum_{j=1}^{n} p_{ijk} = 1 \qquad 1 \le i,k \le n$$

$x_{ij} :=$ control fraction S_j has over S_i.

Notation:
$$P_i = p_{ijk}, \qquad 1 \le j,k \le n$$
$$x_i = [x_{i1}, x_{i2} \cdots x_{in}]^T, \qquad 1 \le i \le n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \implies x_i \in \Delta_1^n$$

$r_{i\ell k} :=$ the resources that S_k can apply with its control over S_ℓ to alter the control fraction for S_i.

General Model.

With the notation just defined we have the general dynamical model.

$$\dot{x}_{ij} = \sum_{l=1}^{n} \sum_{k=1}^{n} r_{ilk} (p_{ijk} - x_{ij}) x_{lk}$$

$$1 \leq i, j \leq n$$

In vector notation,

$$\boxed{\dot{x}_i = \sum_{l=1}^{n} (P_i R_{il} - r_{il} \otimes x_i) x_l}$$

General Model

If $R_{il} = 0$ for $i \neq l$, the General model reduces to a direct sum of naive models.

This means power is localized and uncoupled between pairs of structures.

Example General model, $n=2$.

$$4\times4\quad\left[\begin{aligned}\dot{x}_1 &= (P_1R_{11} - r_{11}\otimes x_1)x_1 + (P_1R_{21} - r_{21}\otimes x_1)x_2\\ \dot{x}_2 &= (P_2R_{21} - r_{21}\otimes x_2)x_1 + (P_2R_{22} - r_{22}\otimes x_2)x_2\end{aligned}\right.$$

P_1, P_2, R_{ij} are two dimensional blocks

Example. General triangular model, $n=3$.

$$9\times9\quad\left[\begin{aligned}\dot{x}_1 &= (P_1R_{11} - r_{11}\otimes x_1)x_1 + (P_1R_{12} - r_{12}\otimes x_1)x_2 + (P_1R_{13} - r_{13}\otimes x_1)x_3\\ \dot{x}_2 &= (P_2R_{22} - r_{22}\otimes x_2)x_2 + (P_2R_{23} - r_{23}\otimes x_2)x_3\\ \dot{x}_3 &= (P_3R_{33} - r_{33}\otimes x_3)x_3\end{aligned}\right.$$

$P_i \quad R_{ij} \quad 3\times3$ blocks

* hierarchical ordering of the structures.

Back to the naive model $\qquad \dot{x} = (PR - r \otimes x)x$

Recall $\quad P = p_{ij} \quad 1 \le i,j \le n \qquad \sum_{i=1}^{n} p_{ij} = 1, \; j = 1 \cdots, n$

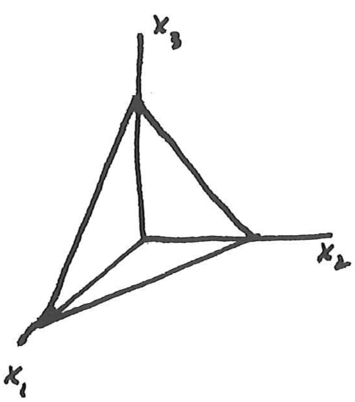
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad p_{ij} \geq 0 \quad 1 \le i,j \le n$

Column stochastic

$R = \mathrm{diag}(r_1 \cdots, r_n) \qquad \underline{\underline{r_i > 0}} \quad i = 1, \cdots, n$

$x(0) \in \Delta_1^n$

$\Rightarrow x(t) \in \Delta_1^n \quad$ for all t.



Equilibria:

Solve $\quad (PR - r \otimes x)x = 0$

$$PRx = \langle r, x \rangle x$$

$$T(x) := \frac{PRx}{\langle r, x \rangle} \qquad T: \Delta_1^n \to \Delta_1^n$$

By fixed point argument $\exists \; x^e \in \Delta_1^n \ni \qquad = x^e$.
Thus there exists at least one equilibrium, x^e.

But then

$$PR\,x^e = \langle r, x^e \rangle\, x^e$$

So the equilibrium is a positive eigenvector of the positive matrix PR and $\langle r, x^e \rangle$ is the corresponding eigenvalue.

For positive matrices we have the Perron-Frobenius theory.

Defn An $n \times n$ matrix A with non negative entries is called irreducible if A is not similar to a matrix of the form

$$\left[\begin{array}{c|c} M & O \\ \hline P & N \end{array}\right]$$

where M and N are square matrices.

Thm (Perron–Frobenius). An $n \times n$ positive matrix A that is irreducible has a simple eigenvalue corresponding to its spectral radius and a corresponding positive eigenvector x^e that is strictly positive. $(x^e >> 0)$

If P and therefore PR is irreducible then the system

$$\dot{x} = (PR - r \otimes x)x$$

has a unique equilibria in the (relative) interior of Δ_1^n.

If P is reducible, then there may be several equilibria.

Theorem 1. The solution of the system $\dot{x} = (PR - r \otimes x)x$ with initial condition $x(0) \in \Delta_1^n$ converges to an equilibrium x^e.

- P is irreducible $\Rightarrow$ x^e is unique
- P is reducible $\Rightarrow$ x^e may depend on the I.C. $x(0)$.

Proof of Theorem 1.

Fact: The solution of the system $\dot{x} = (PR - r \otimes x)x$ with initial condition $x(0) \in \Delta_1^n$ is given by

$$x(t) = \frac{e^{PRt} x(0)}{\langle e^{PRt} x(0), \bar{e} \rangle}$$

where

$$\bar{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Proof of Theorem 1:

Notation: $p_{*j} := j^{th}$ column vector of P

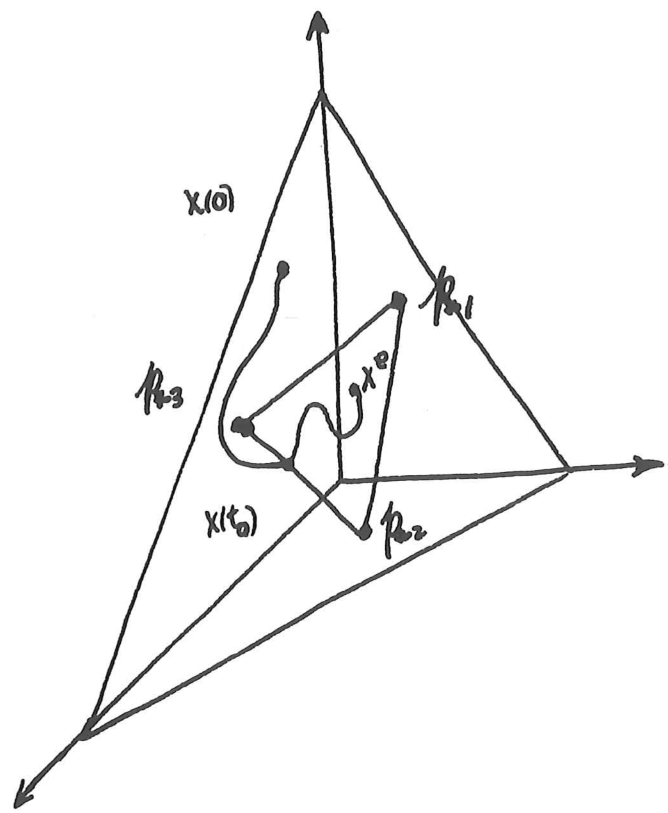
Theorem 2. $x^e \in \operatorname{conv}(p_{*1}, p_{*2}, \cdots, p_{*n}) = C$

Theorem 3. If $x(t)$ is the solution of $\dot{x} = (PR - r \otimes x)x$, $x(0)$. and t_0 is the first time $\ni x(t) \in C$, then

$$x(t) \in C$$

for all $t \geq t_0$.

Simple geometry



$$\dot{x} = (PR - r \circledcirc x)\,x$$

$x(0)$

$p_{*j} :=$ j^{th} column of P.

$x^e :=$ <u>an</u> equilibrium

Socio-Political Interpretations

① If P is reducible we say the system is in a <u>coalition</u>.

② If $p_{*j} = e_j$ for $j \in N \subset \{1, \cdots, n\}$ we say the system is in <u>partial anarchy</u>.

③ If $p_{*j} = e_j$ for $j = 1, \cdots, n$, we say that the system is in <u>pure anarchy</u>. (pure competition alá Hirsch)

④ <u>Revolutions</u> of four types.

1.
2.
3.

$\Big\}$ discontinuous changes in $x(t)$, P, R, respectively.

4. emergence of a new structure.

⑤ <u>Alliances</u>. S_j & S_k make aware their respective profiles to one another.

Example.

$$P = \begin{bmatrix} .7 & .6 & 0 & 0 \\ .3 & .4 & 0 & 0 \\ 0 & 0 & .5 & .8 \\ 0 & 0 & .5 & .2 \end{bmatrix} \qquad R = \mathrm{diag}(1, 1, 1, 1)$$

(a)

$$x(0) = (.1, .2, .3, .4)^T \longrightarrow x^e = (.2, .1, .43, .27)^T$$

$$x(0) = (.0, .0, .1, .9)^T \longrightarrow x^e = (0, 0, .62, .38)^T$$

$$x(0) = (.1, .1, .4, .4)^T \longrightarrow x^e = (.13, .07, .49, .31)^T$$

(b) $R = \mathrm{diag}(1, 1.1, 1, 1)$

$$x^e = (.67, .33, 0, 0)^T \qquad \text{unique!}$$

(c) $R = \mathrm{diag}(1, 1, 1.07, 1)$

$$\bar{x}^e = (0, 0, .61, .39)^T \qquad \text{unique!}$$

Example.

$$P = \begin{bmatrix} .3 & .4 & 0 & 0 \\ .4 & .4 & 0 & 0 \\ .1 & .1 & .6 & .8 \\ .2 & .1 & .4 & .2 \end{bmatrix}$$

$S_3 + S_4$ are in coalition

$R = \mathrm{diag}(1.2, 1.2, 1, 1) \rightarrow x^e = (0, 0, .67, .33)$ unique!

$R = \mathrm{diag}(1.4, 1.4, 1, 1) \rightarrow x^e = (.064, .072, .566, .298)$ unique!

Corollary. If $r_j > \max_{i \neq j} r_i$ and $p_{*j} = e_j$, and $x_j(0) > 0$

then $\lim_{t \to \infty} x_j(t) = e_j$.

A structure that has dominant resources can possess all the control.

Emergence of a new structure.

$$\underset{n\times n}{P} \longrightarrow \underset{(n+1)\times(n+1)}{\hat{P}}$$

$\hat{p}_{*n+1}$ is the new profile vector

r_{n+1} is the corresponding resource.

Survival: (i) If $r_{n+1} > \{r_1 \cdots r_n\}$ S_{n+1} survives

(ii) If $r_{n+1}\hat{p}_{n+1,n+1} > \rho({}_{(n)}\hat{P}\hat{R})$ S_{n+1} survives.

(iii) If $\hat{P}$ is irreducible S_{n+1} survives.

In this case one of the other structures must legitimize S_{n+1} by redefining its profile, say $p_{n+1\,j} > 0$.

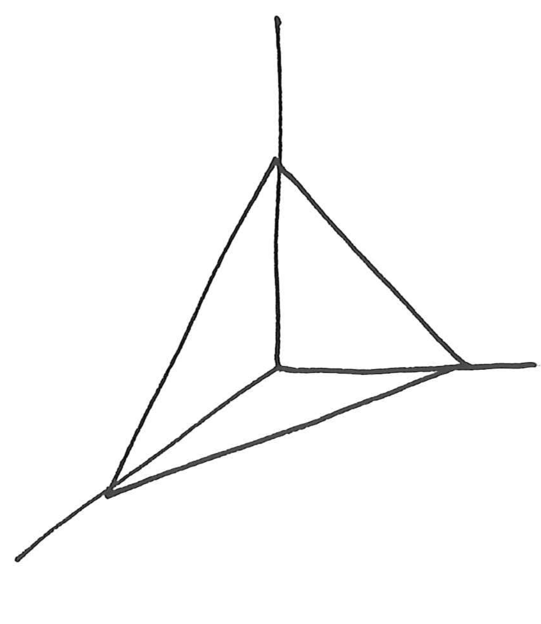
(iv) If the other structures do not legitimize S_{n+1}, extinction is possible, particularly if $\rho({}_{(n)}\hat{P}\hat{R}) = \rho(\hat{P}\hat{R})$.

<u>Theorem</u> 4. (Rate of convergence) If P is irreducible and diag $R >> 0$, then the equilibrium x^e is a <u>sink</u> of the system. Thus there is a $c>0$ so that

$$\| x(t) - x(0) \| \le B e^{-tc} \| x(0) - x^e \| ,$$

for all $x(0) \in \Delta_1^n$.

<u>Theorem</u> 5. (Periodicity of the coefficients yields periodicity of a solution).



Alliances — Power Shifts.

Three questions:

(1) If a structure alters its profile in some way what effect does this have upon the equilibrium?

(2) Can a structure alter its profile in such a way as to cause the equilibrium to move in a specified direction?

(3) What happens to the equilibrium if the resources of a particular structure increase or decrease?

revolutions of the second and third type.

Notation:

$$P = [p_{*1}, p_{*2}, \cdots, p_{*(n-1)}, p_{*n}]$$

$$P' = [p_{*1}, p_{*2}, \cdots, p_{*(n-1)}, p']$$

P is the base profile. P' is the altered profile.

$$x^e := \text{equilibrium of } PR$$

$$x' := \text{equilibrium of } P'R$$

→ Assume now and throughout that P is irreducible & $R \gg 0$.

<u>Theorem 6.</u> Assume $p_{*n} \gg 0$. Let x^e be the equilibrium of PR. Then there is an open neighborhood $\mathcal{O}$ of x^e in the relative topology of Δ_1^n, such that for each $x' \in \mathcal{O}$ there is a unique $p' \in \Delta_1^n$ for which x' is the principal eigenvector of the irreducible

$$P'R = [p_{*1}, \cdots, p_{*(n-1)}, p']R.$$

Given $x' \in \mathcal{O}$. Define $\rho' = \langle r, x' \rangle$

$$p' = \frac{\rho'}{r_n x_n'}\left(x' - \sum_{j=1}^{n-1} \frac{r_j x_j'}{\rho'} p_{*j}\right)$$

<u>Theorem 7.</u>

Now suppose

$$x_1 \longleftrightarrow [p_{*1}, \cdots, p_{*(n-1)}, p_{*n_1}] = P_1$$

$$x_2 \longleftrightarrow [p_{*1}, \cdots, p_{*(n-1)}, p_{*n_2}] = P_2$$

$$\rho_1 = \rho(P_1 R) \qquad \rho_2 = \rho(P_2 R)$$

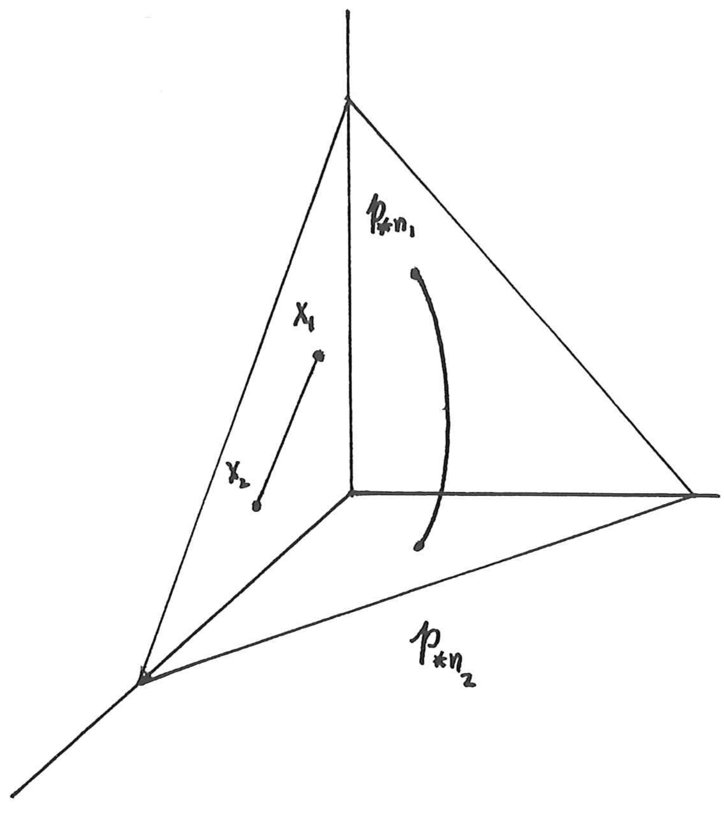
$$x_\mu = \mu x_1 + (1-\mu) x_2 \qquad \rho_\mu = \mu \rho_1 + (1-\mu)\rho_2$$

$$P_\mu = [p_{*1}, \cdots, p_{*(n-1)}, p_\mu]$$

$$p_\mu = \lambda p_{*n_1} + (1-\lambda) p_{*n_2} + \frac{x_{\mu n}}{r_n x_{1n} x_{2n}} \lambda(1-\lambda)(\rho_2 - \rho_1)(x_1 - x_2)$$

$$\lambda = \mu x_{1n} / x_{2n}$$

$$P_\mu R x_\mu = \rho_\mu x_\mu$$


$p_* n_1$
x_1
x_2
$p_* n_2$

<u>Corollary.</u> For a balanced system $(R = cI)$

$$p_\mu = \lambda p_{*n_1} + (1-\lambda) p_{*n_2}$$

where

$$\lambda = \mu x_{1n} / x_{\mu n}$$

Assume $R = I$ (equiv cI)

$$\dot{x} = (PR - r \otimes x) x$$

becomes

$$\dot{x} = (P - \langle e, x \rangle) x$$

$$\dot{x} = (P - I) x$$

<u>linear</u>

What happens to x^e when p_{*n} is changed?

Suppose: $$p' = p_{*n} + \mu v$$ $v \in \Delta_0^n$

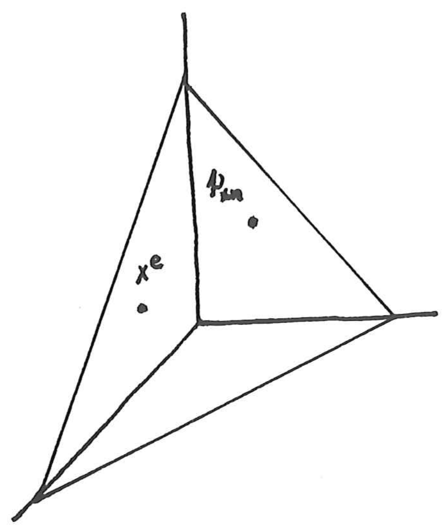
$\sum_1^n v_i = 0$

Then $$x' = f(\mu, v).$$

$$F(x) := (PR - r \otimes x)x$$

$$DF(x^e) = PR - r \otimes x^e - \rho I$$

$$\rho = \rho(PR)$$



Theorem 9. Under the above assumptions, if $(p'R - r \otimes x' - \rho' I)$ is dissipative for $\mu \geq 0$, then the orthogonal projection of $x' - x^e$ to v is positive and increasing.

A system $\dot{x} = (PR - r \circledast x)x$ is said to be <u>conforming</u> if $DF(x^e)$ is dissipative.

What is the relevance of these conditions?

intrinsic aspect of a society. Selecting strategies to increase one's control becomes a complex problem – a political one.

For balanced societies, power functions as one would expect! Namely, any structure can unilaterally increase or decrease the power of any other structure (including itself).

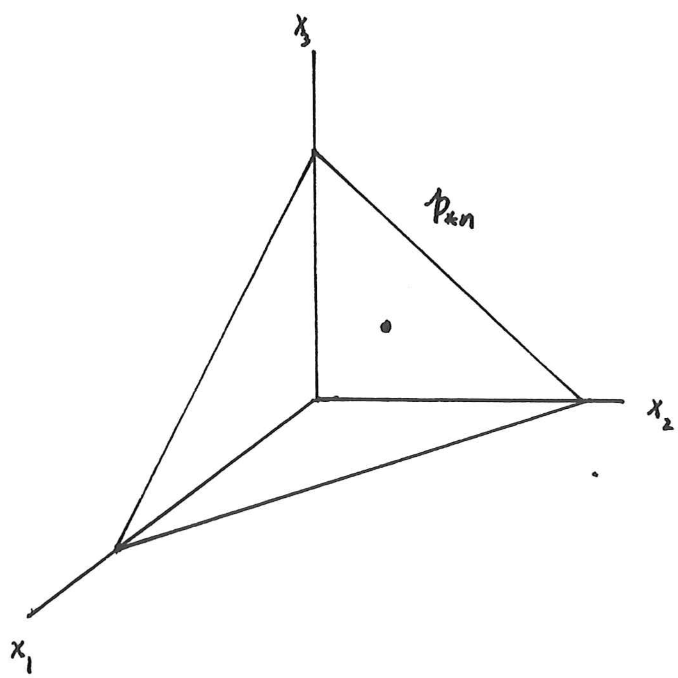
Assume $R = cI$.

<u>Theorem</u> 10. If P is irreducible, then Structure n can increase the value x_k^e for any $k=1,\cdots,n$, by changing its profile vector to

$$p' = p_{*n} + \mu_k e_k + \sum_{\substack{j=1 \\ j \neq k}} \mu_j e_j ,$$

where $p' \in \Delta_1^n$, $\mu_k > 0$ and $\mu_j \leq 0$, $j \neq k$.

Similarly, if $p' \in \Delta_1^n$, $\mu_k < 0$ and $\mu_j \geq 0$ for $j \neq k$, then x_k^e decreases.



Changing the resources. r_n

$$r_n' = r_n + \mu \qquad R' = \mathrm{diag}(r_1, \dots, r_{n-1}, r_n')$$

Theorem 11. $x'(\mu)$ is the principal eigenvector of PR', where P is irreducible. Then

$$\lim_{\mu \to \infty} x'(\mu) = p_{*n}.$$

Theorem 12. Suppose that P is irreducible and $PR' - r \circledast x' - \rho' I$ is dissipative. Then

$$\| x' - p_{*n} \|_2$$

is monotone decreasing with increasing μ, $\mu > 0$.

Arbitration. Introducing binding mediation into the naive model yields an equilibrium completely determined by the arbitor.

Two structures in conflict + mediator $\longrightarrow$ conflict resolution

Underlying assumption: the structures have equal resources.

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad x^e \text{ indeterminant.}$$

Add the mediator:

$$\bar{P} = \begin{bmatrix} p & 0 & p_1 \\ 0 & p & p_2 \\ \varepsilon & \varepsilon & p_3 \end{bmatrix} \qquad \bar{R} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \varepsilon' \end{bmatrix}$$

Result: $$x^e = c \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

<u>Theorem 13.</u> Let $Q = I_{(n-1)}$ and $R = r\,I_{(n-1)}$ denote a system in conflict. The mediator-added model is

$$P = \left[\begin{array}{c|c} pQ & p_{*n} \\ \hline (1-p) \cdots (1-p) & \end{array}\right] \qquad R = \left[\begin{array}{c|c} r\,I_n & 0 \\ \hline 0 & r_n \end{array}\right]$$

The equilibrium x^e of the system $\dot{x} = (PR - r \otimes x)\,x$ is given by

$$ {}_{(n-1)}\bar{x}^{e} = \alpha \; {}_{(n-1)}p_{*n}$$

where $\beta = \alpha(1-p_{nn})$, and where β is the positive solution to

$$rp - r_n + \frac{r_n(1-p_{nn})}{\beta} = \frac{r(1-p)\beta}{1-\beta},$$

so that $0 < \beta < 1$.

Generalized version:

$$PR = \left[\begin{array}{c|c} D & T \\ \hline S & E \end{array}\right]$$

$$\dot{D} = p\, I_k$$

x^e depends on T and E $(+\, S)$

via secondary way.

Some results about the triangular general model.

$$(\ast) \quad \begin{aligned} \dot{x} &= (PR - r \otimes x)x + (PG - g \otimes x)y \\ \dot{y} &= (QT - t \otimes y)y \end{aligned}$$

Assume. P, Q irreducible. $r \gg 0,\ g \gg 0,\ t \gg 0$

---

1) There exists an unique equilibrium to $\ast$:

$x^e \oplus y^e$

$$\begin{bmatrix} PR & PG \\ & QT \end{bmatrix} \begin{bmatrix} x^e \\ y^e \end{bmatrix} = \begin{bmatrix} [\langle r, x^e \rangle + \langle g, y^e \rangle]\, x^e \\ [\ \langle t, y^e \rangle\ ]\, y^e \end{bmatrix}$$

2) For any initial condition $y(0)$,

$$\lim_{t \to \infty} y(t) = y^e.$$

(3) For each $y(t)$,

$$(PR - r \otimes x)x + (PG - g \otimes x)y(t) = 0$$

has a unique solution x^e_t.

(4) For each fixed $y(s)$ the solution to

$$\frac{dx}{dt} = \dot{x} = (PR - r \otimes x)x + (PG - g \otimes x)y(s) \quad \forall\, x(0) \in \Delta^n_1$$

converges to x^e_t. In particular, y^e.

(5)

$$\lim_{t \to \infty} x^e_t = x^e$$

(6) For any initial condition $x(0) \oplus y(0)$, the solution to the system $(\star)$ $x(t) \oplus y(t)$ converges to $x^e \oplus y^e$.

Let $\quad \dot{x} = (PR - r \otimes x)x + (PG - g \otimes x)y$

$\quad \dot{z} = (PR - r \otimes z)z + (PG - g \otimes z)y^e$

Compute $\quad \Delta x := x - z \qquad \Delta y := y - y^e$

$$\dot{\Delta x} = \left[PR - \left(r \otimes z + (\langle g, y\rangle + \langle r, x^e\rangle)I\right]\Delta x\right.$$

$$- (r \otimes \Delta x)\Delta x + PG\,\Delta y - (g \otimes z)\Delta y$$

$\Delta x(0) = 0$

$$\Delta x = \Phi(t)\int_0^t \Phi^{-1}(s)\left[\langle r, \Delta x\rangle \Delta x + PG\,\Delta y - \langle g, \Delta y\rangle z\right] ds$$

$$\|\Delta x\| \leq e^{kt}\left(\|r\|\,\|\Delta x\|^2 + \gamma\|\Delta y\|\right)$$

Tracking Inequality

7. This result generalizes to every triangular general model with irreducible profiles.