Lectures on Compact Čech Cohomology. (1965)

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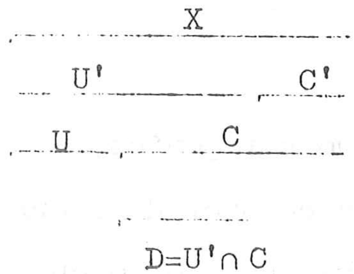
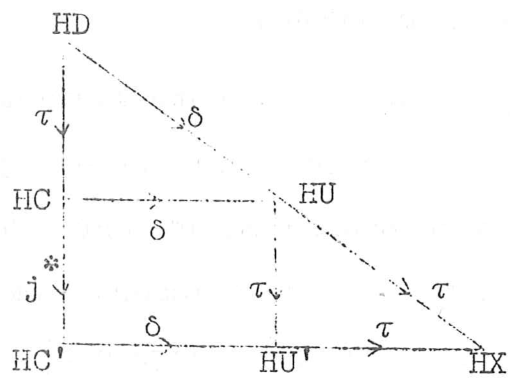
CC2



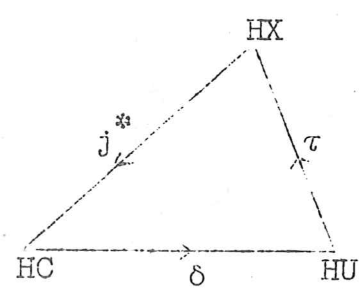
If U is open in U' which in turn is open in X, C = X - U, C' = X - U', D = U' ∩ C, and j is the inclusion : C' → C then the diagram



is commutative.

CC3 Exactness

If U is open in X, C = X - U, and j is the inclusion : C → X, there is an exact triangle



CC4 Dimension

$$H^q(\text{point}) = 0, \ q \neq 0.$$
$$H^0(\text{point}) = \text{coefficient group}.$$

Remarks

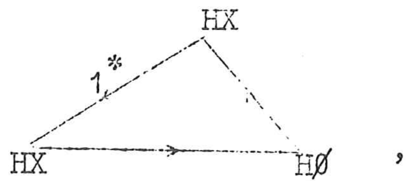
(a) CC2 replaces the usual excision axiom.

(b) We appear to have no homotopy axiom, it has been replaced by the requirement that the groups in the receiving category $\underline{\underline{G}}$ be countable (this requirement is more restrictive, for example it cuts out singular theory). For compact cohomology we shall show that countability implies satisfaction of the homotopy axiom.

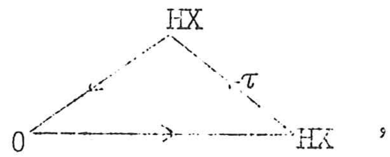
Lemma 1. If $U = X$ then $\tau = 1$.

Proof.

Exactness with $U = \emptyset$ gives $1^* = 1$ (our cohomology is functorial) and therefore $H\emptyset = 0$.



Exactness with $U = X$ now gives



so τ is an isomorphism.

Also CC2 provides $\tau^2 = \tau$, hence $\tau = 1$.

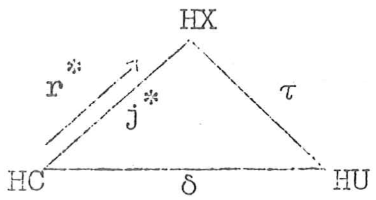
Lemma 2. If U is open in X, $C = X - U$, $r:X \to C$ is a proper retraction and j is the inclusion $C \subset X$, then the sequence

$$0 \to HU \to HX \underset{r^*}{\overset{j^*}{\rightleftarrows}} HC \to 0$$

is split exact.

Proof.

$rj = 1$ implies $j^* r^* = 1$ (our cohomology functor is contravariant) consequently j^* is an epimorphism, and therefore in the exact triangle



$\delta = 0$ as required.

Lemma 3. $HJ = 0$ where $J = [01)$, the half-open interval.

Proof.

We prove $H[ab) = 0$, $b > a > 0$.

Lemma 2 provides exactness of

$$0 \to H[0a) \xrightarrow{\tau'} H[0b) \to H[ab) \to 0$$

so τ' is mono, it is sufficient then to show that τ' is also epi.

For any $t > 0$ we have the exact sequence

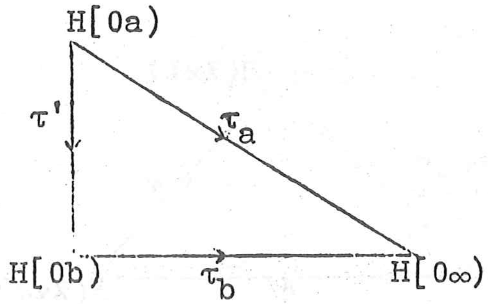
$$0 \to H[0t) \xrightarrow{\tau_t} H[0\infty) \to H[t\infty) \to 0$$

consequently τ_t is mono.

Define (i) $G = \bigcup_{t>0} \tau_t H[0t)$, then $G \subset H[0\infty)$ and therefore must be countable.

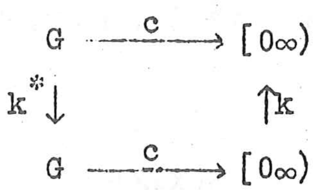
(ii) $c : G \to [0\infty)$ by $c(g) = \text{glb}\{t : g \in \tau_t H[0t)\}$.

Suppose τ' is not epi, then $\exists\, x \in H[0b)$ which does not belong to image τ', let $\tau_b(x) = y$. Axiom CC2 provides the commutative diagram



and we know the three maps are mono, hence $c(y) \geqslant a$ and therefore image $c \neq 0$. It remains then to show image $c = 0$.

If $k:[0\infty) \to [0\infty)$ is the homeomorphism "multiplication by k", $k > 0$, then $k^*:H[0\infty) \to H[0\infty)$ maps G to itself and the diagram



is commutative. Consequently if $s > 0$ belongs to the image of c then so does ks for <u>all</u> $k > 0$, contradicting the countability of G.

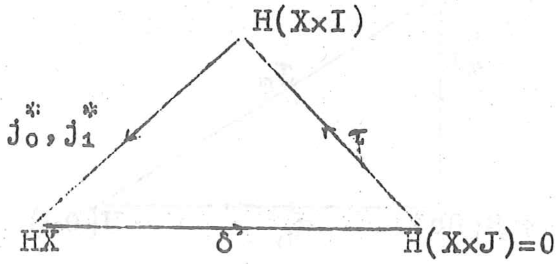
By similar argument, if $X \in \underline{\underline{A}}$ then <u>$H(X \times J) = 0$</u>.

<u>Theorem 1.</u> If $f, g:X \to Y$ are properly homotopic then $f^* = g^*$.

<u>Proof.</u>

Let $j_0, j_1:X \to X \times I$ be the inclusions $X \to X \times 0$, $X \to X \times 1$ respectively, and suppose $r:X \times I \to X$ is a proper retraction.

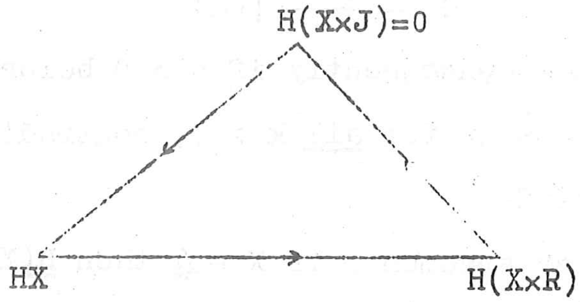
The triangle



is exact consequently j_o^* and j_1^* are both iso. Also $rj_o = 1 = rj_1$, therefore $j_o^* = r^{*-1} = j_1^*$. If now $h:X \times I \to Y$ is a proper homotopy with $hj_o = f$, $hj_1 = g$ then $f^* = j_o^* h^* = j_1^* h^* = g^*$.

<u>Theorem 2.</u> (Suspension) $H^q(X \times R) \cong H^{q-1}X$, $q > 0$.

<u>Proof.</u> The triangle



is exact.

For the remainder of this section we assume that a compact cohomology functor exists which, when restricted to the subcategory whose objects are compact spaces, agrees with the Čech cohomology functor. The existence will be established in section 3. Our present aim is simply to illustrate the calculating power of the exactness axiom by means of a few examples.

A. By induction, applying Theorem 2,

$$H^q E^n = \begin{cases} Z(\text{coeff. gp.}), & q = n \\ 0 & \text{otherwise.} \end{cases}$$

B. Since B^n is proper homotopy equivalent to a point,

$$H^q B^n = \begin{cases} Z, & q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$H^q S^n = \begin{cases} Z, & q = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

For, apply the exactness axiom with $X = B^{n+1}$, $U = E^{n+1}$, $C = S^n$.

C.

$$H^i(CP^n) = \begin{cases} Z & i = 0, \ i \text{ even} \leqslant 2n \\ 0 & \text{otherwise.} \end{cases}$$

Regard E^{2n} as an open subset of CP^n, then its complement will be CP^{n-1}. Proceed by induction, which begins trivially, using exactness in the below tabulation to establish the inductive step.

	HCP^{n-1} ⟵	HCP^n ⟵	HE^{2n}
2n	0	Z	Z
2n − 1	0	0	0
2n − 2	Z	Z	0
⋮	⋮	⋮	⋮
2	Z	Z	0
1	0	0	0
0	Z	Z	0

D. Let T denote the unit tangent bundle of S^n. We shall calculate the cohomology of T.

Now $T \subset S^n \times S^n$, for consider a pair (x, t) consisting of a point $x \in S^n$ and a unit tangent at x. By means of the exponential map, "bend" the tangent so as to give a second point $y \in S^n$, and regard (x, t) as the point (x, y) of $S^n \times S^n$. A little further thought will convince you that

$T \times R = S^n \times S^n - \Delta - \nabla$, where Δ is the diagonal, and ∇ the skew diagonal, of $S^n \times S^n$. We will calculate the cohomology of $T \times R$, and then "desuspend" using Theorem 2. Consider the following table:

$$\Delta \cup \nabla \xleftarrow{\;j^*\;} S^n \times S^n \longleftarrow T \times R$$

	$\Delta \cup \nabla$ (two disjoint copies of S^n)	$S^n \times S^n$	$T \times R$
2n	0	Z	
2n − 1	0	0	
⋮	⋮	⋮	
n + 1	0	0	
n	Z ⊕ Z	Z ⊕ Z	?
n − 1	0	0	
⋮	⋮	⋮	
1	0	0	
0	Z ⊕ Z	Z	

The first two columns we know; to fill in the third we must consider the map j^* (induced from inclusion) in dimensions 0 and n; it is clear straightway that $H^{2n}(T \times R) = Z$.

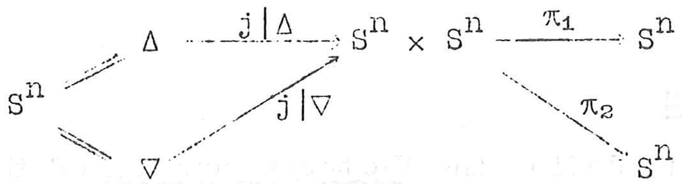
a) <u>dimension 0</u> : let a generate $H^0(S^n \times S^n)$, then $j^*a = \alpha + \beta$ where α, β generate respectively the first and second factors of $H^0(\Delta \cup \nabla)$. (The reader is left to check this assertion). Thus $H^1(T \times R) = \text{Coker } j^* = Z$.

b) <u>dimension n</u> : let a, b generate respectively the first and second factors of $H^n(S^n \times S^n)$. Then

(i) $j^*a = \alpha + \beta$ where α, β generate the first and second factors of $H^n(\Delta \cup \nabla)$.

(ii) if α, β are defined by (i), $j^*b = \alpha + (-)^{n+1}\beta$.

(To check this, use the fact that we have maps



for which $\pi_1 j|\Delta = \pi_2 j|\Delta = \pi_1 j|\nabla =$ identity on S^n, and $\pi_2 j|\nabla =$ antipodal map. The oscillating sign in (ii) is due to the variability of the antipodal map's homotopy class.)

Thus
$$H^n(T \times R) = \text{Ker } j^* = \begin{cases} Z, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

$$H^{n+1}(T \times R) = \text{Coker } j^* = \begin{cases} Z, & n \text{ odd} \\ Z_2, & n \text{ even.} \end{cases}$$

Therefore applying Theorem 2

$$H^0T = Z$$

$$H^{n-1}T = Z\ (0) \text{ for n odd (even)}$$

$$H^nT = Z\ (Z_2) \text{ for n odd (even)}$$

$$H^{2n-1}T = Z$$

and $H^iT = 0$ otherwise.

2. DOWKER'S THEOREM

Let X be a set, define the Victoris complex of the set as follows. A Victoris p-simplex is a finite ordered subset $(x_0, \ldots, x_p)$, $x_i \in X$, possibly with repeats, i.e. a point of X^{p+1}.

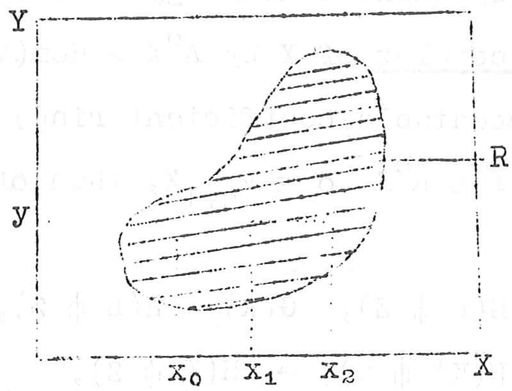
V_pX = free abelian group generated by all p-simplexes,

$VX = \Sigma_{p=0}^{\omega} V_pX$,

$d(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i(x_0, \ldots, \hat{x}_i, \ldots, x_p)$.

Def. A <u>relation</u> between two sets X and Y is a subset of $X \times Y$.

A subset $R \subset X \times Y$ gives rise to two complexes $K \subset VX$, $L \subset VY$ in the following way. A simplex A of VX is in K if and only if A is related to some point of Y, i.e. if $A = (x_0, \ldots, x_p)$ then $A \in K \iff \exists\, y \in Y$ such that $x_i \times y \in R$, $0 \leqslant i \leqslant p$.



Similarly a simplex B of VY belongs to L if and only if B is related to some $x \in X$.

Consider the category $\underline{\underline{R}}$, which has as its objects relations $R \subset X \times Y$, and for which Mor (R, R'), $R \subset X \times Y$, $R' \subset X' \times Y'$, consists of all pairs of maps

$$\begin{cases} \xi : X \to X' \\ \eta : Y \to Y' \end{cases}$$

satisfying $(\xi \times \eta)\, R \subset R'$.

Let $\underline{\underline{H}}$ denote the category of abelian groups and homomorphisms. Then we have two functors F, $G : \underline{\underline{R}} \to \underline{\underline{H}}$ defined by:-

$$F(R) = HK, \quad G(R) = HL,$$

$$F(\xi, \eta) = \xi_* : HK \to HK', \quad G(\xi, \eta) = \eta_* : HL \to HL'.$$

<u>Theorem 3</u> (Dowker)

The functors F and G are naturally isomorphic.

There is a corresponding cohomology version. Define the <u>Alexander cochain complex</u> of X by $A^q X = \mathrm{Hom}(V_q X, Z) = V_q X \pitchfork Z$ (or replace Z by any countable coefficient ring)
$AX = \Sigma_{q=0}^{\infty} A^q X$, and if $f \in A^q X$, $\sigma \in V_{q+1} X$, then $\delta f \sigma = (-)^{q+1} f d\sigma$.
Redefine F, G by

$$F(R) = H(K \pitchfork Z), \quad G(R) = H(L \pitchfork Z),$$

$$F(\xi, \eta) = \xi^* : H(K' \pitchfork Z) \to H(K \pitchfork Z),$$

$$G(\xi, \eta) = \eta^* : H(L' \pitchfork Z) \to H(L \pitchfork Z).$$

Then Theorem 3 remains true.

The remainder of this section is devoted to proving Theorem 3 for homology; the cohomology version then follows at once by application of the Universal Coefficient Theorem.

<u>Def.</u> A <u>double complex</u> $D = \Sigma D_{p,q}$ is a collection of abelian groups $D_{p,q}$, p, q integers ≥ 0, and homomorphisms $d_1 : D_{p,q} \to D_{p-1,q}$, $d_2 : D_{p,q} \to D_{p,q-1}$ such that $d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$.

Lemma 4 Given two complexes K, L we can construct a double complex.

Proof. Define -
$$D_{p,q} = K_p \otimes L_q$$
$$d_1 = d_K \otimes 1$$
$$d_2 = w_K \otimes d_L$$
where w_K is the sign changing "trick" $w_K A_p = (-)^p A_p$. Trivially $d_1^2 = d_2^2 = 0$.

Also $d_2 d_1 (A_p \otimes B_q) = d_2(d_K A_p \otimes B_q) = (-)^{p-1} d_K A_p \otimes d_L B_q$

and $d_1 d_2 (A_p \otimes B_q) = d_1((-)^p A_p \otimes d_L B_q) = (-)^p d_K A_p \otimes d_L B_q$,

therefore $d_1 d_2 + d_2 d_1 = 0$.

Lemma 5 Given a double complex we can make a "single" complex out of it.

Proof. Given $D = \Sigma D_{p,q}$, define -
$$D_n = \sum_{p+q=n} D_{p,q}$$
$$d = d_1 + d_2.$$

Lemma 6 If the double complex D is vertically (or horizontally) exact then $HD = 0$.

Proof. Suppose $H_n D \neq 0$, let $h \neq 0 \in H_n D$ and suppose that $x = \sum_{p+q=n} x_{p,q} \in D_n$ is a representative d-cycle for h.

Define the filtration of x to be $\max\{p : x_{p,q} \neq 0\}$ and choose $x \in h$ to be of minimum filtration, say p_o, arising from x_{p_o,q_o}. Now $dx = 0$, therefore $d_2 x_{p_o,q_o} = 0$; by the vertical exactness hypothesis $x_{p_0,q_0} = d_2 y_{p_0,q_0+1}$. Let $x^* = x - d(y_{p_0,q_0+1})$, then x^* is a representative d-cycle for h, but $\text{filt}(x^*) < \text{filt}(x)$ contradicting our original choice of x.

Def. Point complex, $K_p = \begin{cases} Z, & p = 0 \\ 0, & p \neq 0 \end{cases}$

Call this Z.

Def. Augmentation $\varepsilon : K \to Z$ is defined by mapping each vertex in K to the unit $1 \in Z$, and each simplex of dimension > 0 in K to 0. This is a chain map, i.e. $\varepsilon d = 0 = d\varepsilon$, and therefore induces $\varepsilon_* : HK \to Z$.

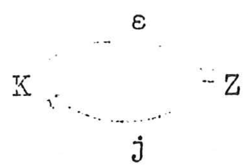
Lemma 7 Assuming K has at least one vertex, and defining $\underline{K}$ by exactness in

$$0 \to \underline{K} \to K \xrightarrow{\varepsilon} Z \to 0$$

then $0 \to H\underline{K} \to HK \xrightarrow{\varepsilon_*} Z \to 0$ is exact.

$H\underline{K}$ is called the reduced homology of K.

Proof.



Define $j : Z \to K$ by mapping the unit of Z to a vertex (one exists by hypothesis) of K. Then $\varepsilon j = 1$, so $\varepsilon_* j_* = 1$, therefore ε_* is epi. Exactness elsewhere in the sequence is obvious.

Lemma 8 If X is non-empty then VX is acyclic.

Proof.

$$v \in VX \underset{j}{\overset{\varepsilon}{\rightleftarrows}} Z$$

Augment VX and choose a vertex v, define j as in lemma 4. Then $\varepsilon j = 1$ and $j\varepsilon$ is chain homotopic to 1 by the chain homotopy $h(x_0, \ldots, x_p) = (v, x_0, \ldots, x_p)$. Therefore ε_* is an isomorphism and from lemma 7 we obtain $H\underline{VX} = 0$.

Lemma 9 $K \otimes Z \cong K$.

Consequently $H(K \otimes Z) \cong HK$.

Proof of Dowker's theorem

Given $R \subset X \times Y$ determining K, L we need an isomorphism $HK \cong HL$.

Define $D \subset K \otimes L$ to be generated by all $A_p \otimes B_q$, $A_p \in K$, $B_q \in L$, with A_p related to B_q. Augment L, then applying lemma 9 $K \otimes L \xrightarrow{1 \otimes \varepsilon} K \otimes Z$ induces a map : $D \to K$ which, since every $A \in K$ is related to some vertex of L, is an epimorphism. Define E to be that subdouble complex of D which is the kernel of $1 \otimes \varepsilon$, therefore $0 \to E \to D \to K \to 0$ is exact. We claim $HE = 0$, obtaining

$$HD \xrightarrow{(1 \otimes \varepsilon)_* \ \cong} HK.$$

Similarly $$HD \xrightarrow{(\varepsilon \otimes 1)_* \ \cong} HL.$$

Denoting the constructed isomorphism by R_*, it is straightforward to check the commutativity of

$$\begin{array}{ccc} HK & \xrightarrow{R_*} & HL \\ \Big\downarrow{\scriptstyle \xi_*} & & \Big\downarrow{\scriptstyle \eta_*} \\ HK' & \xrightarrow[R'_*]{} & HL' \end{array}$$

To show $HE = 0$:-

Given $A \in K$ let RA be the subset of all points of Y which are related to A, then the sequence

$$0 \to E = \sum_{A \in K} A \otimes \underline{VRA} \to D = \sum_{A \in K} A \otimes VRA \to \sum_{A \in K} A \otimes Z = K \to 0$$

is exact.

By lemma 8 VRA is acyclic and $H(\underline{VRA}) = 0$, therefore E is vertically exact and $HE = 0$ by lemma 6.

3. EXISTENCE

(3.1) Construction

Let X be a locally compact Hausdorff space with a countable base. Recall the Victoris chain complex VX, and the Alexander cochain complex AX.

Given $f \in AX$, define the support of f, written S_f, by $x \in S_f \Leftrightarrow$ every neighbourhood of x contains a simplex σ such that $f\sigma \neq 0$. Then S_f is closed.

Given U open in X let $A_U X$ be the set

$$\{f : S_f \text{ is compact and contained in } U\}.$$

Then $A_U X$ is a subcomplex of AX, i.e. $f, g \in A_U X$ implies that $f + g$ and δf both belong to $A_U X$.

Exercise: Check that $S_{f+g} \subset S_f \cup S_g$, $S_{\delta f} \subset S_f$ and that S_{f+g}, $S_{\delta f}$ are both compact.

Note - S_{f+g} may be smaller than $S_f \cup S_g$, viz. $g = -f$, zero having empty support.

We are particularly interested in

$$A_X X = \{f \text{ with compact support}\}$$
$$A_\phi X = \{f \text{ with empty support}\}$$

and define

$$HX = H(A_X X/A_\phi X).$$

The definition gives us at once topological invariance.

A proper map $f:X \to Y$ induces a chain map $VX \xrightarrow{f_*} VY$ and Hom (, Z) gives a corresponding cochain map $AX \xleftarrow{f^*} AY$. Suppose $g \in AY$, then it is easily seen that $S_{gf} = S_{f^*g} \subset f^{-1}S_g$. Thus if g has compact support in Y, $f^{-1}S_g$ will be compact in X and have S_{f^*g} as a closed subset, therefore A_YY is mapped by f^* to A_XX. If g has empty support then so has f^*g and therefore $A_\phi Y$ is mapped to $A_\phi X$. More generally for any open set $V \subset Y$, A_VY is mapped to $A_{f^{-1}V}X$.

We have then associated with f a homomorphism

$$f^*:HY \to HX.$$

Clearly $1^* = 1$ and $(f_1f_2)^* = f_2^*f_1^*$.

(3.2) Connection with Čech Theory.

Before verifying that HX satisfies the compact cohomology axioms, we show it gives precisely the Čech cohomology groups for a compact space.

Let X be compact, then $A_XX = AX$. Thus $HX = H(AX/A_\phi X)$. Let Y be a covering of X by open sets, and define a relation R between X and Y as follows. A point $x \in X$ is related to an open set $U \in Y$ if $x \in U$. Then R determines a subcomplex of both VX and VY, denote these respectively by V^RX, V^RY. Applying the cohomology version of Dowker's theorem, we have

$$H(V^RX \pitchfork Z) \cong H(V^RY \pitchfork Z).$$

The right hand side is the cohomology of the nerve of Y.
To put the left hand side in a more convenient form, define $A^R X$ as the annihilator of $V^R X$ in AX. We have an exact sequence of free abelian chain complexes

$$0 \rightarrow V^R X \rightarrow VX \rightarrow VX/V^R X \rightarrow 0,$$

and (since $V^R X$ is a subcomplex of VX) $\pitchfork$ Z gives the following exact sequence of cochain complexes

$$0 \leftarrow AX/A^R X \leftarrow AX \leftarrow A^R X \leftarrow 0.$$

Therefore

$$H(AX/A^R X) \cong \text{cohomology of the nerve of } Y.$$

Consider now all coverings of X directed by refinement, then we have two direct systems of groups. Taking direct limits

$$\varinjlim H(AX/A^R X) \cong \text{Čech cohomology of } X.$$

But

$$\begin{aligned}\varinjlim H(AX/A^R X) &= H(\varinjlim AX/A^R X) \quad \text{since direct limit is exact}\\ &= H(AX/\textstyle\bigcup A^R X)\\ &= H(AX/A_\phi X) \quad \text{as required.}\end{aligned}$$

Exercise : check that $\bigcup A^R X = A_\phi X$.

(3.3) Exactness, the homomorphisms δ, τ.

Lemma 10 If U is open in X, then restriction induces an isomorphism

$$H(A_U X/A_\phi X) \cong H(A_U U/A_\phi U) = HU.$$

Proof. The inclusion $VU \to VX$ is mono, so Hom (, Z) produces an epimorphism $\lambda : AX \to AU$, $\lambda(f) = f|U$. Now λ maps $A_U X$ onto $A_U U$. If $f \in A_U X$ then $S_{f|U} = S_f$, compact and contained in U; further, given $g \in A_U U$, extend it to $\bar{g} \in A_U X$ by defining

$$\bar{g}\sigma = g\sigma, \ \sigma \in VU$$
$$= 0 \ , \ \sigma \notin VU,$$

then $S\bar{g}$ = closure in X of $Sg = Sg$, compact. Obviously λ also maps $A_\phi X$ to $A_\phi U$. Therefore we have an epimorphism $\lambda : A_U X / A_\phi X \to A_U U / A_\phi U$. This map is also mono, - for given $F \in \text{Ker } \lambda$, represent F by f in $A_U X$, then $S_{f|U}$ is empty. We require S_f empty, suppose $x \in S_f$, then $x \in U$, therefore $\exists$ a neighbourhood N of x in X such that $N \subset U$ and $f|N = 0$, contradiction. (We used U open in X only to assert N open in X).

Lemma 11 If C is closed in X, U = X - C, then restriction induces an isomorphism

$$H(A_X X / A_U X) \cong H(A_C C / A_\phi C) = HC.$$

Before proving Lemma 11 we deduce:-

Theorem 4 This cohomology functor satisfies our exactness axiom (CC3).

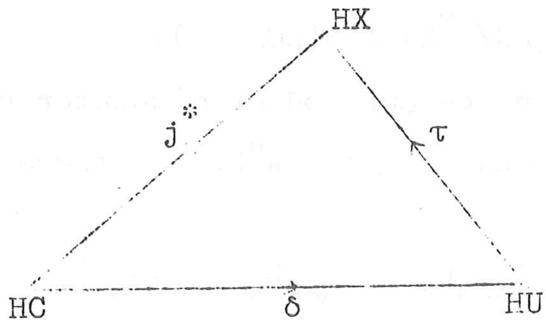
Proof. We have cochain complexes

$$A_\phi X \subset A_U X \subset A_X X$$

and therefore an exact sequence

$$0 \to A_U X / A_\phi X \to A_X X / A_\phi X \to A_X X / A_U X \to 0.$$

Taking cohomology, lemmas 10 and 11 provide an exact triangle



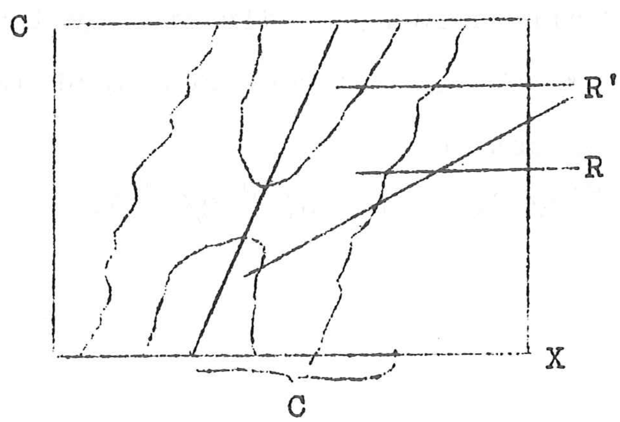
Proof of Lemma 11.

Proceed as for Lemma 10 to produce an epimorphism

$$\mu : A_X X / A_U X \to A_C C / A_\phi C, \quad \mu(f) = f|C.$$

Alas this is not mono. For example let $\{x_n\}$ be a sequence of points of $X - C$ with limit point $x_0 \in C$, take f as the 0-cochain which maps each x_i, $i \neq 0$, to 1 and the rest of X to zero. Then $x_0 \in S_f \cap C$, consequently $f \notin A_U X$; however, $f|C$ clearly has empty support.

We need $H(A_X X/A_U X) \overset{\mu^*}{\cong} H(A_C C/A_\phi C)$. Let R be a neighbourhood of the diagonal $\Delta = \{(x, x) : x \in C\}$ of $X \times C$.



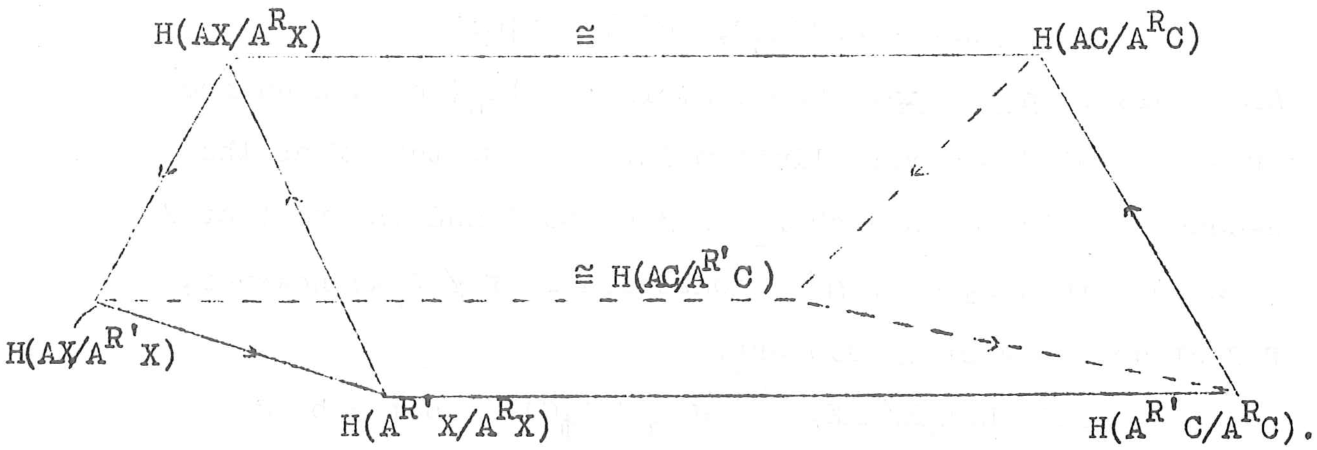
Then R is a relation between X and C. Using the notation introduced in (3.2), we have from Dowker's theorem

$$H(AX/A^{R}X) \cong H(AC/A^{R}C).$$

Now let $R' \subset R$ be a neighbourhood of a counter compact subset of Δ in $X \times C$. Then $AX \supset A^{R'}X \supset A^{R}X$, and therefore we have an exact sequence

$$0 \to A^{R'}X/A^{R}X \to AX/A^{R}X \to AX/A^{R'}X \to 0.$$

A similar sequence exists for C. Taking cohomology produces the triangles



Therefore $H(A^{R'}X/A^{R}X) \cong H(A^{R'}C/A^{R}C)$.

Consider the set of all pairs R, R' directed by inclusion; by the above procedure we obtain a direct system of isomorphisms. Take direct limits to obtain

$$\varinjlim H(A^{R'}X/A^{R}X) \cong \varinjlim H(A^{R'}C/A^{R}C).$$

But LHS $= H(\varinjlim A^{R'}X/A^{R}X) = H(\cup A^{R'}X/\cup A^{R}X)$

and RHS $= H(\varinjlim A^{R'}C/A^{R}C) = H(\cup A^{R'}C/\cup A^{R}C)$.

Our proof is completed by

<u>Sublemma 1</u> a) $\cup A^{R}X = A_{U}X$ b) $\cup A^{R'}X = A_{X}X$

c) $\cup A^{R}C = A_{\phi}C$ d) $\cup A^{R'}C = A_{C}C$

We leave this proof to the reader.

<u>Sublemma 2</u> The isomorphism we have produced is indeed induced from restriction.

<u>Proof.</u> It is enough to show that the isomorphisms

$$\varinjlim H(AX/A^{R}X) \cong^{\alpha} \varinjlim H(AC/A^{R}C)$$
$$\varinjlim H(AX/A^{R'}X) \cong^{\beta} \varinjlim H(AC/A^{R'}C)$$

are induced from restriction. We deal with the first, the second follows in a similar manner. Consider R as a subset of $X \times X$ $(R \subset X \times C \subset X \times X)$, then it determines two complexes which we denote by $AX/(A^{R}X)_1$ and $AX/(A^{R}X)_2$. The <u>naturality</u> of the Dowker isomorphisms produces a commutative diagram

$$\begin{array}{ccc} H(AX/(A^{R}X)_1) & \cong & H(AC/A^{R}C) \\ \uparrow 1^{*} & & \uparrow \\ H(AX/(A^{R}X)_1) & \cong & H(AX/(A^{R}X)_2). \end{array}$$

Taking direct limits this becomes

$$\begin{array}{ccc} H(AX/A_{U}X) & \overset{\alpha}{\cong} & H(AC/A_{\phi}C) \\ \uparrow 1^{*} & & \uparrow \mu^{*} \\ H(AX/A_{U}X) & \underset{1^{*}}{\cong} & H(AX/A_{U}X) \end{array}$$

as required.

(3.4) Exercise: Check the commutativity axioms CC1, CC2 and the dimension axiom CC4.

(3.5) Countability

It remains to show that our graded group HX is countable. We shall need:

Theorem 5 (Continuity) Let $\{V_i\}$ be a collection of open sets of X which is closed under finite union, and $V = \cup V_i$. Then $HV = \varinjlim HV_i$.

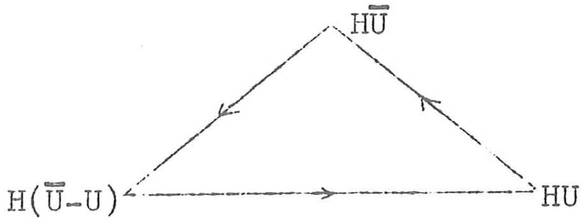
Proof.

$$\varinjlim HV_i = \varinjlim H(A_{V_i}X/A_{\phi}X) \text{ by Lemma 10}$$
$$= H(\varinjlim A_{V_i}X/A_{\phi}X)$$
$$= H(\cup A_{V_i}X/A_{\phi}X).$$

Therefore if we can show $\cup A_{V_i}X = A_V X$, the proof may be completed by a further application of Lemma 10. It is obvious that $\cup A_{V_i}X \subset A_V X$. Conversely, given $f \in A_V X$, S_f is covered by a finite number of the V_i since it is compact and contained in V. Let V_* denote the union of those finitely many open sets; then V_* belongs to our collection, and $f \in A_{V_*}X$.

For a compact space with countable base the Čech cohomology groups, with countable coefficients, are countable. (Compactness and countable base together ensure that in producing the Čech groups one may use a countable directed set of finite open coverings. Clearly the cohomology groups of the nerve of a finite covering are finitely generated, and therefore countable.) Thus, by (3.2), if X is compact, HX is countable.

Suppose now X is locally compact with a countable base. Then X is expressible as the union of a countable family of open sets, each of whose closures is compact. Let U be such a set, with compact closure $\overline{U}$. Then $H\overline{U}$ and $H(\overline{U} - U)$ are countable by compactness; also the triangle



is exact, thus HU is countable.

Application of Theorem 5 now completes the proof that HX is countable.

4. RING STRUCTURE

(4.1) Cup Product

We first give AX a ring structure - if $f^p \in A^pX$, $f^q \in A^qX$ define their cup product by

$$f^p \cup f^q(x_0,\ldots,x_{p+q}) = (-)^{pq}\, f^p(x_0,\ldots,x_p).f^q(x_p,\ldots,x_{p+q})$$

Remarks

(a) This product is associative, it is neither commutative nor skew commutative, but by the time we reach cohomology level it becomes skew commutative.

(b) The sign convention ensures "naturality" in applications.

Lemma 12 $\quad S_{f \cup f'} \subset S_f \cap S_{f'}$.

Cor. $A_X X$ and $A_\phi X$ are ideals in AX. Hence we have an induced associative ring structure on $A_X X/A_\phi X$.

Lemma 13 $\quad \delta(f^p \cup f^q) = (\delta f^p \cup f^q) + (-)^p\, (f^p \cup \delta f^q)$.

(Recall $\delta f^p(x_0, \ldots, x_{p+1}) = (-)^{p+1} f^p d(x_0, \ldots, x_{p+1})$ from section 2).

Cor. If Z and B are the subcomplexes of $A_X X/A_\phi X$ consisting of cocycles and coboundaries respectively, then B is an ideal in Z.

Proof. Given $z \in Z$ and $b \in B$, there exists f such that $b = \delta f$, from lemma 2

$$\delta(z \cup f) = \pm (z \cup b),$$

hence $z \cup b \in B$ as required.

Therefore there is an induced ring structure on HX = Z/B.

<u>Theorem 6</u> (skew commutativity)

If $\xi \in H^pX$, $\eta \in H^qX$ then $\xi\eta = (-)^{pq}\eta\xi$.

(At cohomology level we leave out the $\cup$).

<u>Proof.</u>

Let $r:VX \to VX$ be the automorphism given by $r(x_0, \ldots, x_p) = (x_p, x_{p-1}, \ldots, x_0)$; r is <u>not</u> a chain map i.e. $rd \neq dr$. Let $s:VX \to VX$ be the automorphism given by

$$s(x_0, \ldots, x_p) = (-)^{\frac{1}{2}p(p+1)} (x_0, \ldots, x_p).$$

<u>Lemma 14</u> Let $\rho = rs:VX \to VX$, then ρ is a chain map.

<u>Def.</u> A <u>carrier</u> $\Gamma:K \to L$ assigns to each simplex $\sigma \in K$ a subcomplex $\Gamma\sigma \subset L$ such that if σ is a face of σ' then $\Gamma\sigma \subset \Gamma\sigma'$. A carrier will be called <u>acyclic</u> if each $\Gamma\sigma$ is acyclic. $f:K \to L$ is <u>carried</u> by Γ if, for each $\sigma \in K$, $f\sigma \subset \Gamma\sigma$.

<u>Lemma 15</u> (see Hilton and Wylie p.111)

Two chain maps carried by the same acyclic carrier are chain homotopic.

Now ρ and 1 are both carried by $\sigma \to V\sigma$ and therefore are chain homotopic. Hence $\rho^* = 1:HX \to HX$. Represent ξ, η by cocyles f, g then

$$r(f \cup g) = rg \cup rf$$

$$\rho(f \cup g) = (-)^{\lambda}\rho g \cup \rho f$$

where $\lambda = \frac{1}{2}p(p + 1) + \frac{1}{2}q(q + 1) + \frac{1}{2}(p + q)(p + q + 1) = pq \bmod 2$.

$$\rho^*(\xi\eta) = (-)^{pq}(\rho^*\xi)(\rho^*\eta),$$ completing the proof.

Application 1. There is no orientation reversing homeomorphism of CP^2.

Proof. In section 1 we calculated the compact cohomology of CP^2 to be $H^0 = H^2 = H^4 = Z$, $H^i = 0$ all other i. Let these three non-zero groups have generators 1, x, y respectively; then $y = x^2$ (two complex lines meet in a point). Suppose h is a homeomorphism of CP^2, then h^* must map generators to generators, thus $h^* x = \pm x$, when from the ring structure $h^* x^2 = x^2$.

Therefore h must be orientation preserving.

(4.2) A generator for $H^n E^n$.

We know $H^n E^n = Z$; suppose ξ is a generator, then ξ is representable by a cocycle $f^n \in \Lambda E^n$ with non-empty compact support. Take coordinates $x_1, \ldots, x_n$. Let A_j be the $\frac{1}{2}E^n$ given by $x_j \geq 0$, and let a_j be the 0-cochain which maps points of A_j to 1 and all other points to zero.

Define $$f^n = \delta a_1 \cup \delta a_2 \cup \ldots \cup \delta a_n.$$

Note: $Sa_j = A_j$, non compact.

$S_{\delta a_j} = \partial A_j = \{x : x_j = 0\}$, non compact.

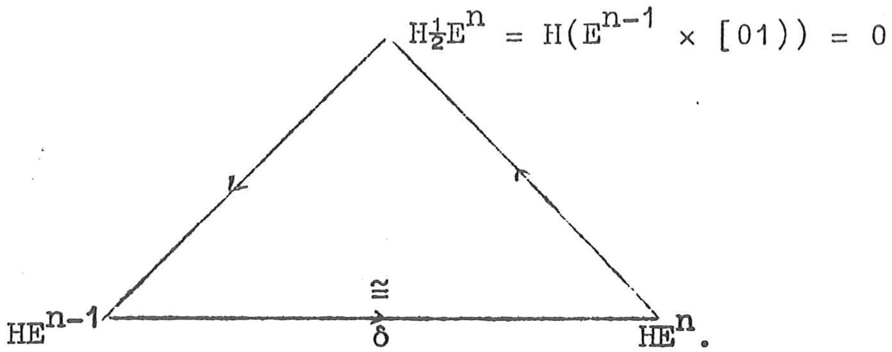
However, by lemma 1,

$$S_{f^n} \subset \bigcap_{j=1}^{n} Sa_j = \text{origin}.$$

Check: $S_{f^n} = \text{origin}$.

<u>Application 2</u> The cohomology class of f^n generates $H^n E^n$.

<u>Proof.</u> By induction on n, which begins trivially with n = 0. Assume the result for n - 1 and consider E^{n-1} as a closed subset of $\frac{1}{2}E^n$, with complement E^n. We have an exact triangle



Therefore δ maps f^{n-1} to a generator of $H^n E^n$. Extend f^{n-1} to g on $\frac{1}{2}E^n$ by $g = f^{n-1} \cup a_n$. Then $\delta g = f^{n-1} \cup \delta a_n$ (neglecting signs), and restriction to E^n completes the proof.

<u>References</u>

1. C.H. Dowker, Homology Groups of Relations, Ann. Math., (56) 1952, 84-95.
2. E.C. Zeeman, Dihomology I, Proc. Lond. Math. Soc., (12) 1962, 609-638.

Application 2. The [illegible] holds [illegible]

Proof. By induction on n, [illegible] trivially [illegible]

Assume the result for [illegible] and consider E^{n-1} [illegible]

subset of E^n, with [illegible]

[illegible]

Therefore [illegible]

[illegible]

and [illegible]

References

1. [illegible]

   [illegible]

2. E.C. Zeeman, [illegible] Proc. Lond. Math. Soc. [illegible]

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