BOSTON UNIVERSITY

DYNAMICAL SYSTEMS INSTITUTE

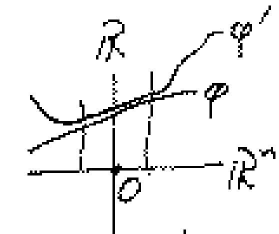
Research Conference in dynamics #1.
June 30 – July 3, 1991

CHAOS AND CATASTROPHES:

Sir Christopher Zeeman.

Catastrophe Theory. (Zeeman)

Classification Theorem



Definition of germ: Let $\varphi, \varphi' : \mathbb{R}^n \to \mathbb{R}$ be C^∞-function.
Call them locally ~~equivalent~~ equal if they agree on nbd 0.
This is an equiv. relation. Let $f = [\varphi]$, the equivalence class.
Call f the germ of φ at 0.

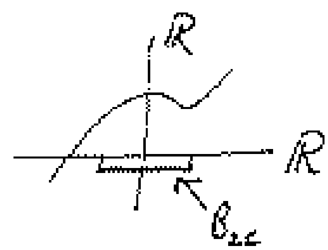
Def Let E = ring of germs at 0 of all C^∞-function $\mathbb{R}^n \to \mathbb{R}$.
$= \infty$-dim vector space.

Note: $f + f' = [\varphi + \varphi']$, $\varphi \in f$, $\varphi' \in f'$, (indep. choice of φ, φ')
$ff' = [\varphi\varphi']$.
$1 = [1]$.

Def Let m = ideal of germs vanishing at 0.

Lemma! m = the (unique) maximal ideal.

Proof Let I be any other ideal $\neq E$.
Claim $I \subset m$. Suppose not. $\therefore \exists f \in I - m$
Claim $\exists \frac{1}{f} \in E$. $\therefore \frac{1}{f} f \in I$ $\therefore 1 \in I$ $\therefore I = E$ contra.
$\therefore I \subset m$ $\therefore m$ maximal.



Proof Choose $\varphi \in f$, $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi 0 \neq 0$
Choose $\varepsilon > 0$ st. $\varphi \neq 0$ on the open ball $B_{2\varepsilon}$ of $\begin{cases} \text{radius } 2\varepsilon \\ \text{centre } 0. \end{cases}$

Construct diffeo $g : \mathbb{R}^n \to B_{2\varepsilon}$ keeping B_ε pointwise fixed.
$(r, \theta) \longmapsto (\rho(r), \theta)$, $r \geq 0$, $\theta \in S^{n-1}$

Let $\varphi' = \varphi g$ $\therefore [\varphi'] = [\varphi] = f$

$$\rho(r) = \begin{cases} r, & 0 \leq r \leq \varepsilon \\ \varepsilon + (r - \varepsilon)\left(1 - e^{\frac{\varepsilon}{r-\varepsilon}}\right), & r \geq \varepsilon \end{cases}$$

$\varphi' \neq 0$ anywhere.
$\therefore \frac{1}{\varphi'} \exists$ $\therefore \exists \frac{1}{f} = \left[\frac{1}{\varphi'}\right]$
as required.

<u>Notation</u> Given $f_1, \dots, f_p \in E$ let $(f_1, \dots, f_p)_E$ = ideal of E generated by f_i's
$= \left\{ \sum_{i=1}^{p} e_i f_i \mid e_i \in E \right\}$.

<u>Lemma 2</u> $(f_1, \dots, f_p)_E (g_1, \dots, g_q)_E = (f_1 g_1, \dots, f_i g_j, \dots, f_p g_q)_E$.

<u>Proof</u> Straightforward.

<u>Note</u>: All definitions so far are coordinate free.
Now choose coords $x_1, \dots, x_n$ for $\mathbb{R}^n$.

<u>Lemma 3</u> $m = (x_1, \dots, x_n)_E$.

<u>Proof</u> $x_i \in m \quad \therefore (x_1, \dots, x_n)_E \subset m$.

Conversely $f \in m \Rightarrow fx = [f(tx)]_0^1 = \int_0^1 \frac{d}{dt}(f(tx))\,dt$

$$= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx)\, x_i \, dt$$

$$= \sum e_i x_i, \quad \text{where } e_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt$$

$\therefore m \subset (x_1, \dots, x_n)_E$.

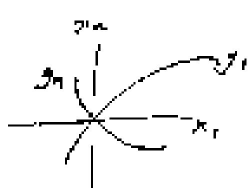
<u>WARNING</u> we are using x_i ambiguously to denote $\begin{cases} i^{th} \text{ coordinate of } x \\ \text{function } \mathbb{R}^n \to \mathbb{R} \\ \text{germ of that function} \end{cases}$

& f ambiguously to denote $\begin{cases} \text{germ} \\ \text{function in that germ} \end{cases}$

<u>Definition</u> Given $f \in E$ define the <u>Jacobian ideal</u> J of f by $J = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

<u>Lemma 4</u> J independent of choice of coordinates.

<u>Proof</u> Let $y_1, \dots, y_n$ be another choice of coords.



$$\therefore \frac{\partial f}{\partial y_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial y_j} \in J$$

$\therefore J^{(y)} \subset J$. Similarly $J \subset J^{(y)}$ $\therefore J = J^{(y)}$

Definition Call E/m^{k+1} the space of k-jets.

Let j^k denote the projection
$$E \longrightarrow E/m^{k+1}$$
$$f \longmapsto j^k f.$$

Call $j^k f$ the k-jet of f.

Remark 1 The above definition is coordinate free.

Remark 2 Take coordinates & expand in Taylor series

$$f = \underbrace{f_0 + f_1 + f_2 + \cdots\cdots + f_k}_{\text{This is the k-jet}} + \underbrace{f_{k+1} + \cdots}_{\text{Call this the k-tail}}$$

const, linear term, quadratic term

This is because all monomials of degree $\geq k+1$ lie in m^{k+1}.

Remark 3. $f_0 + \cdots + f_k$ is coordinate free (being the k-jet), but f_k is not coordinate free.

Example Let $f = y = x + x^2$ be a coordinate change (in the nbd of 0)

$\therefore\ j^2 f = f = y = x + x^2$, independent of choice of x or y,

but $f_2 = \begin{cases} x^2 & \text{in } x\text{-coordinates} \\ 0 & \text{in } y\text{-coordinates} \end{cases}$

Let $\mathcal{G}$ = group of germs at 0 of C^∞ diffeos $\mathbb{R}^n, 0 \longrightarrow \mathbb{R}^n, 0$.
(as before a germ is the equivalence class of a diffeo under the equivalence relation of agreeing on some nbd of 0)

The $\mathcal{G}$ acts on the right of E by composition $\mathbb{R}^n \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$, $\underset{fg}{\longrightarrow}$

$$E \times \mathcal{G} \longrightarrow E$$
$$(f, g) \longmapsto fg$$

Definition Write $f \sim f'$ if $f = f'g$ some $g \in \mathcal{G}$.
This is an equivalence relation.

Definition Call f <u>k-determinate</u> if $\forall f'$, $j^k f = j^k f' \Rightarrow f \sim f'$

ALGEBRA GEOMETRY

Call f <u>determinate</u> if k-det, some k.

Define <u>determinacy</u> of f = least k such that f k-det.

<u>Theorem</u> $m^k \subset mJ \xRightarrow{\mathrm{I}} f$ k-det $\xRightarrow{\mathrm{II}} m^{k+1} \subset mJ$

ALG GEOM ALG

We have slotted the geometric condition in between two algebraic tests.

For example $\begin{cases} f \text{ might be some standard polynomial} \\ f' \text{ might be some piece of applied maths,} \end{cases}$

& then the algebraic test allows us to screw f onto f', in other words to choose coordinates intrinsic to the problem, with respect to which the applied maths will have polynomial form. In other words if the first few terms of a Taylor series in an applied problem satisfy an easily computable algebraic test, this gives us permission to snip off the k-tail rigorously, & maintain all qualitative properties (qualitative = invariant under diffeomorphism)

<u>Example 1</u> $f = x^2$.

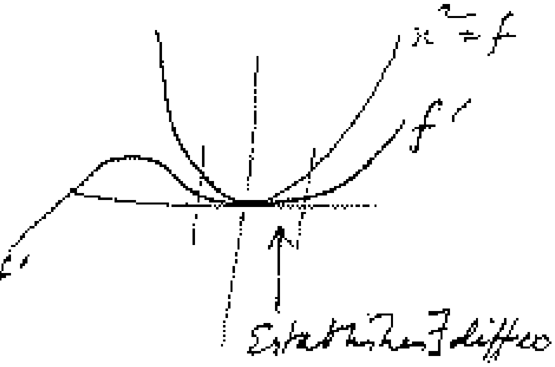
$J = (2x) = (x) = m$

$\therefore mJ = m^2$.

$\therefore$ 2-det.

$\therefore f = x^2 \sim x^2 + ax^3 + bx^4 + \cdots = f'$

$a, b, \ldots$ arbitrary.



Ex 2. $f = x^4$, cusp germ.

$J = (4x^3) = (x^3) = m^3$

$\therefore mJ = m^4$

$\therefore$ 4-det

Ex 3 Elliptic umbilic. $\frac{x^3}{3} - xy^2$.

$J = (x^2 - y^2, -2xy) = (x^2 - y^2, xy)$

$\therefore mJ = (x, y)(x^2 - y^2, xy)$

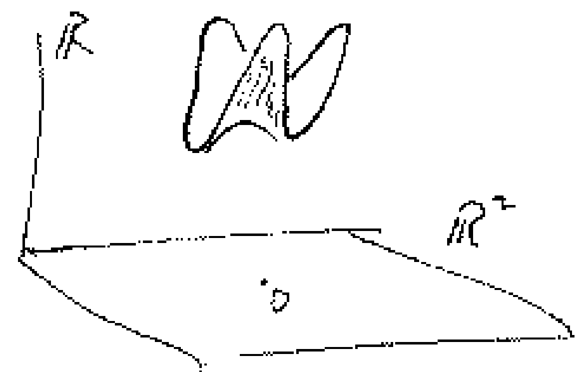
$= (x^3 - xy^2, \cancel{x^2y} - y^3, x^2y, xy^2)$

(by subtracting other generators)

$= (x^3, x^2y, xy^2, y^3)$

$= m^3$.

$\therefore$ 3-det.



Monkey saddle.

$\therefore$ any perturbation of f by terms of degree ≥ 4 produces another monkey saddle, which, in a nbd. of 0, can be mapped onto the first by screwing $\mathbb{R}^2$ onto itself suitably.

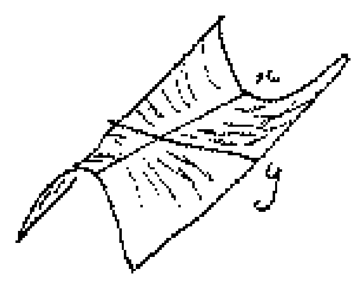
Ex 4 xy^2 indet

$J = (y^2, 2xy) = ym$

$\therefore mJ = ym^2 \not\supset x^k$, $\forall k$.

$\therefore \not\supset m^k$, $\forall k$.

$\therefore$ not det



Note. By an arbitrarily small perturbation, for instance by adding $\varepsilon e^{-\frac{1}{|x|}}$ in $\frac{1}{x}$, we can change the critical...

Lemma 4' $\left.\begin{array}{l} f \text{ } k\text{-det} \\ f \sim f' \end{array}\right) \Rightarrow f' \text{ } k\text{-det}$

Proof. Suppose $j^k f' = j^k f''$. We want to show $f' \sim f''$.

We know $f = f'g$ some $g \in \mathcal{G}$.

Now $f' - f'' \in m^{k+1}$ $\therefore f' - f'' = \sum a_j \mu_j$, where μ_j denote the monomials of degree k in $x_1, \dots, x_n$, which generate m^{k+1} by Lemma 2.

$\therefore \mu_j g$ is a monomial in $g_1, \dots, g_n$, the coordinates of g, which lie in m since $g0 = 0$.

$\therefore \mu_j g \in m^{k+1}$.

$\therefore (f' - f'')g = \sum a_j (\mu_j g) \in m^{k+1}$.

$\therefore j^k((f' - f'')g) = 0$.

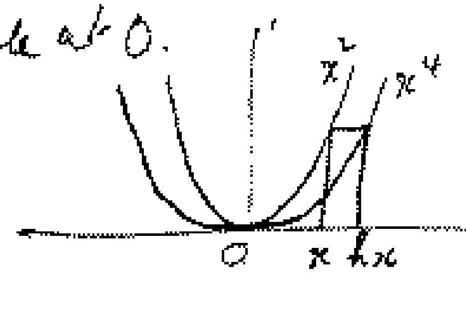
$\therefore j^k(f) = j^k(f'g) = j^k(f''g)$

$\therefore f \sim f''g$ since f k-det

$\therefore f' \sim f \sim f''g \sim f''$ $\therefore$ $\underline{f' \sim f''}$ as required.

Corollary 2 $x^2 \not\sim x^4$. because $\det(x^2) = 2$, $\det(x^4) = 4$

Remark $\exists$ homeomorphism h s.t. $x^2 = x^4 h$, but h cannot be differentiable at 0.



Nakayama's Lemma 5

$$\left.\begin{array}{l} A \text{ ring with unit} \\ \mathfrak{a} \text{ the unique maximal ideal} \\ M, N \text{ modules over } A \\ \quad \text{(contained in some larger module)} \\ M \text{ finitely generated} \\ M \subset N + \mathfrak{a}M \end{array}\right\} \Rightarrow M \subset N.$$

Proof Case ① $N = 0$.

Given $M \subset \mathfrak{a}M$ we have to show $M = 0$.

Suppose not.
Let $g_1, \dots, g_s$ be a minimal set of generators of M.
Then $s \geq 1$ since we are assuming $M \neq 0$.
$g_s \in M \subset \mathfrak{a}M$ $\therefore g_s = \sum_{i=1}^{s} a_i g_i$, $a_i \in \mathfrak{a}$

$$\therefore (1 - a_s) g_s = \sum_{i=1}^{s-1} a_i g_i.$$

Claim $1 - a_s \notin \mathfrak{a}$, otherwise since $a_s \in \mathfrak{a}$, $1 \in \mathfrak{a}$, $\therefore \mathfrak{a} = A$ contra.
$\therefore (1 - a_s)A$ is ideal $\not\subset \mathfrak{a}$.
$\therefore (1 - a_s)A = A$ since $\mathfrak{a}$ maximal.
$\therefore \exists (1 - a_s)^{-1} \in A$ $\therefore g_s = \sum_{i=1}^{s-1} (1 - a_s)^{-1} a_i g_i$

$\therefore g_1, \dots, g_{s-1}$ generate M, contradicting the minimality. $\therefore M = 0$.

Case ②. If $M \subset N + \mathfrak{a}M$ then $\frac{M+N}{N} \subset \frac{N}{N} + \mathfrak{a}\frac{M+N}{N} = 0 + \mathfrak{a}\left(\frac{M+N}{N}\right)$
$\therefore \frac{M+N}{N} = 0$ by Case ①. $\therefore M + N = N$. $\therefore M \subset N$

Beginning of proof of Theorem I

Given $\left\{ \begin{array}{l} \mathfrak{m}^k \subset \mathfrak{m} J \\ j^k f = j^k f' \end{array} \right\}$ we have to show $f \sim f'$.

The technique will be to gradually screw f onto f'

Let $F : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be the homotopy from f to f' given by

$$F(t,x) \equiv F^t(x) = (1-t)f + tf' = f + t(f'-f), \quad t \in \mathbb{R}.$$

In particular $\quad F(0,x) = f$
$\qquad\qquad\quad\ \ F(1,x) = f'$

Let I = unit interval $[0,1]$.

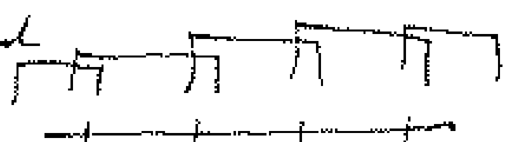
Lemma 6 $\forall t_0 \in I$, $\exists$ germ G at $(t_0, 0)$ of a C^∞ map $\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$
$(t,x) \longmapsto G^t(x)$

such that
(1) $G^{t_0} = 1$ (starts at identity)
(2) $G^t(0) = 0$ (fixes the origin)
(3) $F^t G^t = F^{t_0}$ (screws F^t onto F^{t_0})

Proof that Lemma 6 $\Longrightarrow$ Theorem I (We postpone the proof of Lemma 6)

Diffeomorphism germs form an open subset of the space of map germs $\mathbb{R}^n_0 \to \mathbb{R}^n_0$, because the condition to be a diffeo germ is $\left|\frac{\partial G_i}{\partial x_j}\right| \neq 0$, an open condition.

$G^{t_0} = 1$ a diffeo.
$\therefore G^t$ = diffeo for t sufficiently near t_0
$\therefore F^t \sim F^{t_0}$ for " " "
$\therefore$ for each t_0, $\exists$ nbd, within which all F^t's are equivalent.
Cover I by a finite number of such neighbourhoods, by compactness, & ...



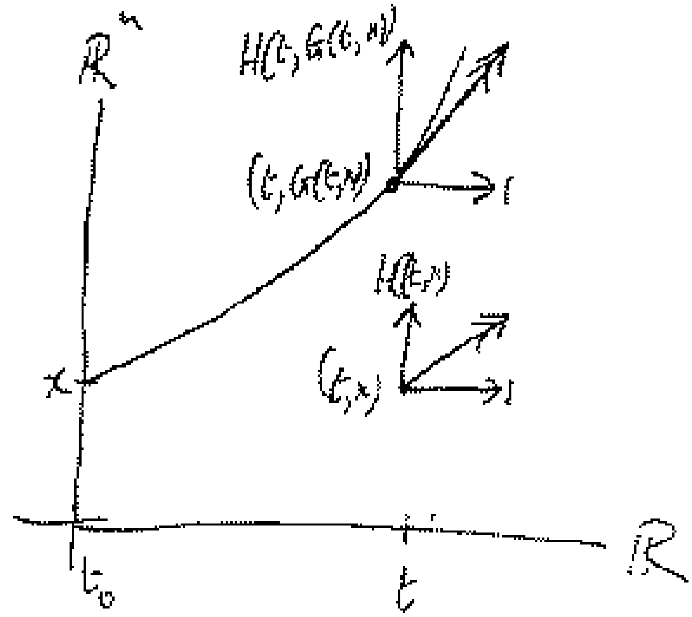
Lemma 7 $\forall t_0 \in I$, $\exists$ germ H at $(t_0, 0)$ of a C^∞ map $\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$

such that (4) $H(t,0) = 0$

(5) $\dfrac{\partial F}{\partial t} + \displaystyle\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} H_i = 0.$

where H_i denotes the i^{th} coordinate of H

Proof that Lemma 7 $\Rightarrow$ Lemma 6 (we postpone the proof of Lemma 7)



Let $G(t,x)$ be the solution of

$$\dot{x} = H(t,x)$$

with initial condition $G(t_0,x) = x$.

The latter $\Rightarrow$ (1). ✓

(4) $\Rightarrow$ t-axis a solution

$\Rightarrow$ (2) ✓

(5) evaluated at $(t, G(t,x))$ gives

$$\frac{\partial F}{\partial t}(t, G(t,x)) + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t, G(t,x)) \underbrace{H_i(t, G(t,x))}_{\frac{\partial G_i}{\partial t}(t,x)} = 0$$

since G satisfies $\dot{x} = H$.

$$\therefore \quad \frac{d}{dt}\left(F(t, G(t,x))\right) = 0.$$

Integrate:

$$\begin{aligned} F(t, G(t,x)) &= \text{const. w.r.t. } t \\ &= F(t_0, G(t_0,x)) \\ &= F(t_0, x) \quad \text{by (1)} \end{aligned}$$

$\therefore \; F^t G^t = F^{t_0}$, namely (3) ✓

Let A = ring of germs at $(t_0, 0)$ of C^∞-functions $\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$.

Let a = maximal ideal of those vanishing at $(t_0, 0)$.

The projection $\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ induces $E \subset A$ and $m \subset a$

Let $\Omega = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right)_A$.

<u>Lemma 8</u> $m^k \subset mJ \Rightarrow m^k \subset m\Omega$.

<u>Remark</u> Notice change of style from ANALYSIS to ALGEBRA

<u>Proof that Lemma 8 ⇒ Lemma 7</u> (we postpone the proof of Lemma 8)

$\frac{\partial F}{\partial t} = f'-f \in m^{k+1} \subset m^k \subset m\Omega$ by Lemma 8.

$\therefore \frac{\partial F}{\partial t} = \sum_j m_j \omega_j$, a finite sum, where $m_j \in m$, $\omega_j \in \Omega$

$\therefore \omega_j = \sum_i a_{ji} \frac{\partial F}{\partial x_i}$

$= \sum_{ji} m_j a_{ji} \frac{\partial F}{\partial x_i}$

$= -\sum_i \frac{\partial F}{\partial x_i} H_i$, putting $H_i = -\sum_j m_j a_{ji}$

Hence (5) ✓

Each m_j vanishes along the t-axis, $\mathbb{R}\times 0 \subset \mathbb{R}\times\mathbb{R}^n$.

$\therefore$ each H_i does also $\therefore H(t,0) = 0$ Hence (4) ✓

<u>Proof of Lemma 8</u> $F = f + t(f'-f)$ $\therefore f = F + t(f - f')$

$$\therefore \frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i} + \underbrace{\underset{\in A}{t}\,\frac{\partial}{\partial x_i}\underset{\in m^{k+1}}{(f-f')}}_{\in m^k} \quad \in \Omega + Am^k.$$

$\therefore J \subset \Omega + Am^k$.

$\therefore m^k \subset mJ \subset m\Omega + mAm^k \subset m\Omega + a(Am^k)$, since $m \subset a$

hypothesis

$\therefore Am^k \subset A(m\Omega + a(Am^k)) = m\Omega + a(Am^k)$.

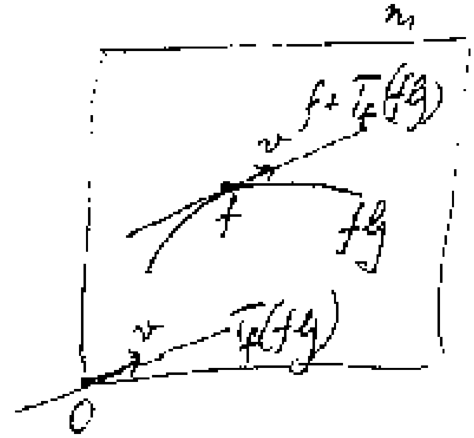
$\therefore Am^k \subset m\Omega$ by Nakayama Lemma 5, since Am^k is finitely generated by monomials of degree k.

$\therefore m^k \subset Am^k \subset m\Omega$

Remark We have proven Lemma 4 $\Rightarrow$ Lemma 1 $\Rightarrow$ Lemma 0 $\Rightarrow$ Theorem I.

There remains to prove Theorem II.

Lemma 9. The tangent plane $T_f(f\mathcal{G}) = mJ$



Proof Given $v \in mJ$, write $v = \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i}$, $\mu_i \in m$

Assembling the μ_i gives $\mu : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$

Let $g^t = 1 + t\mu : \mathbb{R}^n, 0 \to$ itself.

g^t = diffeo for t suff small, since diffeos open in the space of maps.

$\therefore$ g^t is a path in $\mathcal{G}$ from 1

$\therefore$ fg^t is a path in $f\mathcal{G}$ from f.

The tangent to this path $= \frac{\partial}{\partial t}(fg^t) = \sum \frac{\partial f}{\partial x_i} \cdot \frac{\partial g_i^t}{\partial t} = \sum \frac{\partial f}{\partial x_i} \mu_i = v$

$v \in T_f(f\mathcal{G})$ $\therefore$ $mJ \subset T_f(f\mathcal{G})$.

Conversely all such paths cover a nbd. of 1 in $\mathcal{G}$.

$\therefore$ all such paths fg^t cover a nbd. of f in $f\mathcal{G}$. $\therefore$ $T_f(f\mathcal{G}) \subset mJ$

Proof of Theorem II k-det $\Rightarrow$ $\forall f'$, $j^k f = j^k f' \Rightarrow f \sim f'$

$\Rightarrow \{f' \mid j^k f = j^k f'\} \subset \{f' \mid f \sim f'\}$

$\Rightarrow f + m^{k+1} \subset f\mathcal{G}$.

$\Rightarrow T_f(f + m^{k+1}) \subset T_f(f\mathcal{G})$

$\Rightarrow m^{k+1} \subset mJ$, by Lemma 9.

UNFOLDINGS

Given germ $f \in m^2$. $\therefore J \subset m$.

Defn Control space $C = m/J$.

Codimension f = dim C = dim m/J.

Unfolding $\tilde{f} : C \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$(c, x) \longmapsto fx + cx.$

Example 1 Cusp germ $f = x^4$. $\therefore J = (x^3) = m^3$, $\therefore mJ = m^4$, $\therefore$ 4-det not 3-det. $\therefore$ det = 3

As an ideal of E m has one generator x.

As a vector space over $\mathbb{R}$ m has base: $x, x^2, x^3, x^4, \ldots$ (coordinates)

The subspace J " " : $x^3, x^4, \ldots$

$\therefore$ quotient sp m/J " " x, x^2 $\therefore$ codim $f = 2$

Let m/J have coords a, b wrt base x, x^2.

$\therefore c \in m/J$ can be written $c = ax + bx^2$.

$\therefore$ unfolding $\tilde{f} = f + c = x^4 + ax + bx^2$

WARNING x is used ambiguously as $\begin{cases} \text{point of } \mathbb{R} \\ \text{function } \mathbb{R} \to \mathbb{R} \\ \text{germ of that fn} \\ \text{class} \in m/J \end{cases}$

Example 2 Hyperbolic umbilic germ $f = \frac{x^3 + y^3}{3}$. $\therefore J = (x^2, y^2)$ $\therefore mJ = m^3$ $\therefore$ 3-det, not 2-det. $\therefore$ det = 3

$\therefore$ base m/J = x, y, xy.

$\therefore$ codim $f = 3$

$\therefore$ Unfolding $\tilde{f} = f + c = \frac{x^3 + y^3}{3} + ax + by + cxy$

Example 3 Elliptic umbilic germ $f = \frac{x^3}{3} - xy^2$. $\therefore J = (x^2 - y^2, xy)$

$\therefore mJ = (x^3 - xy^2, x^2y, xy^2) = m^3$. $\therefore$ 3-det, not 2-det $\therefore$ det = 3

Base $m/J = x, y, x^2 + y^2$

$\therefore$ codim = 3

Lemma 10 Dimension of jet space. $\dim E/m^{k+1} = \dfrac{n+k!}{n!\,k!}$

Proof If $n=0$ then $E=R$, $m=0$, $m^k=0$. $\therefore$ dim $=1$, $\forall k$. $\therefore$ true

If $k=0$ then $E/m = R$ $\therefore$ dim $=1$, $\forall n$. $\therefore$ true.

Assume true for $n+k<q$.

Give $n+k=q$, $n,k>0$. Then

$$\dim(E/m^{k+1}) = \#\text{ monomials of degree } 0,\dots,k$$

$$= (\#\text{ mons. of degree } 0,\dots,k \text{ in } x_1,\dots,x_{n-1})$$

$$+\ x_n(\#\text{ mons of degree } 0,\dots,k-1 \text{ in } x_1,\dots,x_n)$$

$$= \frac{n-1+k!}{n-1!\,k!} + \frac{n+k-1}{n!\,k-1} = \frac{n+k!}{n!\,k!}$$

Lemma 11 f det $\Leftrightarrow$ codim f finite.

Proof $\Rightarrow$ f det $\Rightarrow$ f k-det, some k

$\Rightarrow m^{k+1} \subset mJ$ by Theorem II.

$\Rightarrow \underbrace{m^{k+1} \subset mJ \subset \underbrace{J \subset}_{} m \subset E}_{\text{finite by Lemma 10}}$

$\therefore$ finite.

$\Leftarrow$ Suppose codim f finite. Now $m \supset m^2+J \supset m^3+J \supset \dots \supset J$

$\therefore m/J \supset \dfrac{m^2+J}{J} \supset \dots \supset J/J = 0$

Since dim m/J finite, sequence can only descend finite # steps.

$\therefore \exists k$ $\dfrac{m^{k-1}+J}{J} = \dfrac{m^k+J}{J}$.

$\therefore m^{k-1} \subset m^k + J$. $\therefore m^{k-1} \subset J$ by Nakayama Lemma 5.

Lemma 12 f det $\Rightarrow \dim \frac{J}{mJ} = n$

Proof Given $g \in J$, write $g = \sum_{i=1}^{n} e_i \frac{\partial f}{\partial x_i}$, $e_i \in E$

Let $e_i = r_i + m_i$, $r_i \in \mathbb{R}$, $m_i \in m$.

$\therefore \sum m_i \frac{\partial f}{\partial x_i} \in mJ$.

$\therefore g = \sum r_i \frac{\partial f}{\partial x_i} \mod mJ$

$\therefore \frac{\partial f}{\partial x_i}$ span J/mJ

$\therefore \dim \frac{J}{mJ} \leq n$.

Suppose $\dim \frac{J}{mJ} < n$. $\therefore \frac{\partial f}{\partial x_i}$ lin dep $/\mathbb{R}$ in J/mJ.

$\therefore \exists r_i \in \mathbb{R}$, not all zero, s.t. $\sum r_i \frac{\partial f}{\partial x_i} = 0$ in J/mJ

$\therefore \sum r_i \frac{\partial f}{\partial x_i} \in mJ$

$\therefore \qquad = \sum m_i \frac{\partial f}{\partial x_i}$, $m_i \in m$

$\therefore \sum (r_i - m_i) \frac{\partial f}{\partial x_i} = 0$

Let v = vector field $\sum (r_i - m_i) \frac{\partial}{\partial x_i}$, which is non-zero at 0

Choose coords $y_1, \ldots, y_n$ st $\frac{\partial}{\partial y_1} = v$

$\therefore \frac{\partial f}{\partial y_1} = 0$.

$\therefore f = f(y_2, \ldots, y_n)$, indep y_1.

$\therefore \forall k$, $y_1^k \notin J$.

$\therefore \forall k$, $m^k \not\subset J$

$\therefore m^{k+1} \not\subset mJ$

$\therefore \forall k$, not k-det

$\therefore f$ not det. Contra

$\therefore \dim J/mJ = n$.

Lemma 13 $f \in m^3 \Longrightarrow \operatorname{codim} f \geq \frac{1}{2}n(n+1)$

Proof Base $m = \overbrace{x_1, \dots, x_n}^{n}, \overbrace{x_1^2, x_1x_2, \dots, x_n^2}^{\frac{1}{2}n(n+1)}, \overbrace{x_1^3, \dots}^{\infty}$

Base $J = \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$, terms degree $\geq$

$\therefore \dim m/J \geq n + \frac{1}{2}n(n+1) - n = \underline{\underline{\frac{1}{2}n(n+1)}}$

Cor 1 $\left.\begin{matrix} f \in m^3 \\ n \geq 3 \end{matrix}\right) \Longrightarrow \operatorname{codim} f \geq 6.$

Cor 2 $\left.\begin{matrix} f \in m^3 \\ \operatorname{codim} \leq 4 \end{matrix}\right) \Longrightarrow n \leq 2.$ Proof $n \not\geq 3$

Splitting Lemma 14

$\left.\begin{matrix} f \in m^2 \\ r = \operatorname{rank} j^2 f \\ r + s = n \\ f \text{ } k\text{-det}, k \geq 3 \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} \exists \text{ separation of variables:} \\ f \sim \mu + e, \text{ where} \\ \mu = \underline{\text{Morse part}} = \pm x_1^2 \pm x_2^2 \pm \dots \pm x_r^2 \\ e = \underline{\text{essential part}} = \text{polynomial} \\ \text{in the } s \text{ variables } x_{r+1}, \dots, x_n \\ \text{of degrees} \geq 3 \text{ \& } \leq k. \end{matrix}\right.$

(cont free)

Example $f = x^2 + 2xy^2$ (variables not separated yet & $\mathbb{R}$ is not Morse, while given it is.)

Put $\xi = x + y^2$

$\therefore f = (x^2 + 2xy^2 + y^4) - y^4 = \xi^2 - y^4 = \text{Morse} + \text{ess}$

Proof of splitting lemma

Choose coords $y_1, \ldots, y_n$

Expand in Taylor series $f = \underbrace{\cancel{f_0} + \cancel{f_1}}_{\text{since } f \in m^2} + f_2 + \cdots$

f_2 = quadratic form $\sum_{ij} \lambda_{ij} y_i y_j$, where (λ_{ij}) symm, $n \times n$, rank

$\exists$ linear change of coordinates $y \to z$ such that w.r.t. z

f_2 has diagonal matrix $\left(\begin{array}{ccc|c} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_r & \\ \hline & 0 & & 0 \end{array}\right)$, $a_i \neq 0$.

$\therefore f_2 = \sum_{i=1}^{r} a_i z_i^2$

Change coords $x_i = \sqrt{|a_i|}\, z_i$. $\therefore f_2 = \pm x_1^2 \pm x_2^2 \cdots \pm x_r^2$
$= \mu$ say.

$\therefore$ we have eliminated $x_{r+1}, \ldots, x_n$ from f_2.

We shall now chuck $x_1, \ldots, x_r$ out of $f_3, f_4, \ldots, f_k$.

Suppose, inductively, we have chucked x_1 out of
$f_3, \ldots, f_{q-1}$, where $q \geq 3$.

We show how to chuck it out of f_q.

Write $f_q = A + 2x_1 B$, where $A \not\ni x_1$, degree $A = q$
degree $B = q-1$

Put $y_1 = x_1 \pm B$, the sign as the sign $\pm$ of x_1^2.

$\therefore \pm y_1^2 = \pm x_1^2 + 2x_1 B \pm B^2$.

Degree $B^2 = (q-1)^2 \geq 2(q-1)$ since $q \geq 3$
$= q + (q-2)$
$\geq q+1$ since $q \geq 3$.

Subst y_1 for x_1, & we've kicked y_1 out of f_q.

Inductively kick $x_1 \ldots x_r$ out of $f_3 \ldots f_k$

Corollary 1 $\left.\begin{array}{l} f \in m^2 \\ f = \mu + e, \text{ as above} \\ \mu = \text{Morse} \\ e \in m^3 \end{array}\right\} \Rightarrow$ Catastrophe space $C_f = C_e$
codim f = codim e
unfolding $\tilde{f} = \mu + \tilde{e}$

Proof $\quad \frac{\partial}{\partial x_i}(\mu + e) = \begin{cases} \pm x_i, & i \leq r \\ \frac{\partial e}{\partial x_i}, & i > r \end{cases}$

Base m = $x_1, \ldots, x_r, x_{r+1}, \ldots, x_n, \ldots$

Base J = $x_1, \ldots, x_r, \qquad \frac{\partial e}{\partial x_1} \ldots$

$\therefore C_f = m/J = \dfrac{(x_{r+1}, \ldots, x_n)_{\mathcal{E}}}{\left\{\frac{\partial e}{\partial x_i}, i > r\right\}} = C_e.$

$\therefore$ codim f = dim C_f = dim C_e = codim e.

$\tilde{f} = \mu + \tilde{e}$.

Corollary 2 $\left.\begin{array}{l} f \in m^2 \\ \text{codim} f \leq 4 \\ f = \mu + e \end{array}\right\} \Rightarrow \leq 2$ essential variables.

Classification

It suffices to examine only the essential part in s variables where $s \leq 2$. $\therefore$ assume $f \in m^3$

$s=0$ Morse.

$s=1$ Cuspoid. $f = a_k x^k + a_{k+1} x^{k+1} + \dots$, $a_k \neq 0$, k.

$\sim a_k x^k$, because $a_k x^k$ is k-det

$\sim \pm x^k$.

$J = (x^{k-1}) = m^{k-1}$.

Base $m = x, x^2, \dots\dots$

Base $J = x^{k-1}, x^{k-2}, \dots$

$\therefore$ Base $m/J = x, x^2, \dots x^{k-2}$. $\therefore$ codim $f = k$.

$\therefore$ Unfolding $f = \pm x^k + a_1 x + a_2 x^2 + \dots + a_{k-2} x^{k-2}$.

k	codim	name.
3	1	fold
4	2	cusp
5	3	swallowtail
6	4	butterfly

$s=2$ umbilics. Suppose $f = ax^3 + bx^2y + cxy^2 + dy^3$

$= a(x-\alpha y)(x-\beta y)(x-\gamma y)$

where α, β, γ are roots.

Lemma. $\exists$ 4 cases, each of which can be reduced by a linear change of coordinates to standard form.

① Distinct real roots $f \sim \frac{x^3}{3} - xy^2$, elliptic umbilic

② 2 complex roots & 1 real. $f \sim \frac{x^3+y^3}{3}$, hyp. "

③ 2 roots equal (: real). $f \sim x^2y$ par. "

Cases ①② are 3-det, & each codim 3.

Case ③ not det; but can make 4-det by $x^2y \pm y^4$, codim 4, parabolic umbilic

Case ④ not det, but can make 4-det by $x^3 \pm y^4$, codim 5 called E_6.

Proof Cases ① & ②. Put $x = \xi + \alpha y$. $\therefore x - \alpha y = \xi$.

$\therefore f = \xi(A\xi^2 + B\xi y + Cy^2)$.

$C \neq 0$ otherwise 2 roots equal $\xi = 0$.

Put $y = \lambda\xi + \eta$. $\therefore f = \xi(A\xi^2 + B\xi(\lambda\xi + \eta) + C(\lambda\xi + \eta)^2)$

Coeff of $\xi\eta$ $= B + 2C\lambda = 0$ if $\lambda = -\frac{B}{2C}$.

$\therefore f = D\xi^3 + E\xi\eta^2$.

$D \neq 0$ otherwise 2 roots equal $\eta = 0$.

$E \neq 0$ $\quad$ " $\quad$ 3 $\quad$ " $\quad$ " $\quad$ $\xi = 0$.

Put $D^{1/3}\xi = x$. $\therefore D\xi^3 = x^3$. $\therefore f = x^3 + Fxy^2$.

Put $|F|^{1/2}\eta = y$ $\quad \therefore Fy^2 = \pm y^2$ $\therefore f = x^3 \pm xy^2$.

Case ①$_-$: 3 distinct real roots $\quad x^3 - xy^2 \sim \underline{\underline{\frac{x^3}{3} - xy^2}}$.

Case ①$_+$: 2 cx & 1 real $\quad x^3 + xy^2 \sim 2x^3 + 6xy^2$

$= (x+y)^3 + (x-y)^3$

$\sim x^3 + y^3$

$\sim \underline{\underline{\frac{x^3 + y^3}{3}}}$

Case ③ $\quad f = a(x - \alpha y)^2(x - \beta y) = a\,\xi^2\eta$, $\quad \xi = x - \alpha y$, $\eta = x - \beta y$

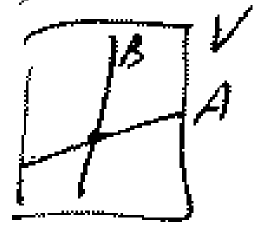
$\sim \underline{\underline{x^2y}}$

Case ④ $\quad f = a(x - \alpha y)^3 = a\xi^3$, $\quad \xi = x - \alpha y$

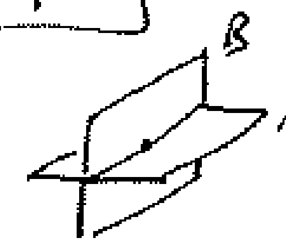
$\sim \underline{\underline{x^3}}$

TRANSVERSALITY

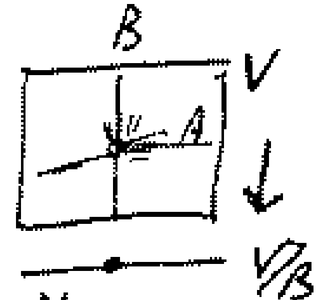
Def ① Given vector space $V \supset$ vector subspaces A, B
we say A transverse to B, written $A \pitchfork B$, if $V = A + B$



Remark if $A \cap B = 0$ then $\dim V = \dim A + \dim B$
if $A \cap B \neq 0$ " " $>$



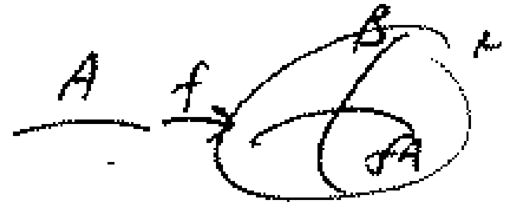
Example Given $V \supset B$.
Choose base for B, & extend to base for V.
Let A = subspace spanned by base els $\notin B$.
Then $V = A + B$ ∴ $A \pitchfork B$.
Further projection $V \to V/B$ maps $A \cong V/B$.
∴ $V \cong V/B + B$.



By abuse of notation, we sometimes write $V = V/B + B$.

Def ② Given manifold $M \supset$ submanifolds A, B
we say $A \pitchfork B$ if $\forall x \in A \cap B$, $T_xA \pitchfork T_xB \subset T_xM$

Def ③ Given $f: A \longrightarrow M \supset B$
we say $f \pitchfork B$ if $fA \pitchfork B$.



Remark Transversality is an open dense condition in the sense that
i) if $\pitchfork$, then suff. small perturbations remain $\pitchfork$
ii) if not, then $\exists$ arb. small " that are.

CODIMENSION

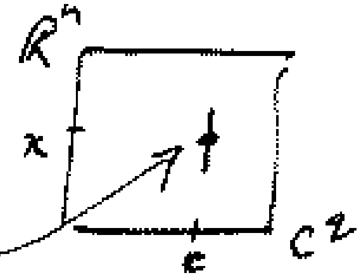
Defn 1. Given vector space $V \supset$ subspace B,

define $$\operatorname{codim}(B \subset V) = \begin{cases} \dim V - \dim B, & \text{if } \dim V \text{ finite} \\ \dim V/B, & \text{if } \dim V\ \infty. \end{cases}$$

Defn 2 Given manifold $M \supset$ submanifold B

define $$\operatorname{codim}(B \subset M) = \begin{cases} \dim M - \dim B, & \text{if } \dim M \text{ finite} \\ \dim {T_xM}/{T_xB}, \ x \in B, & \text{if } \ldots \\ \text{least } \dim A, & \text{st: } \exists f : A \to M,\ f \pitchfork B. \end{cases}$$

LOCALISATION



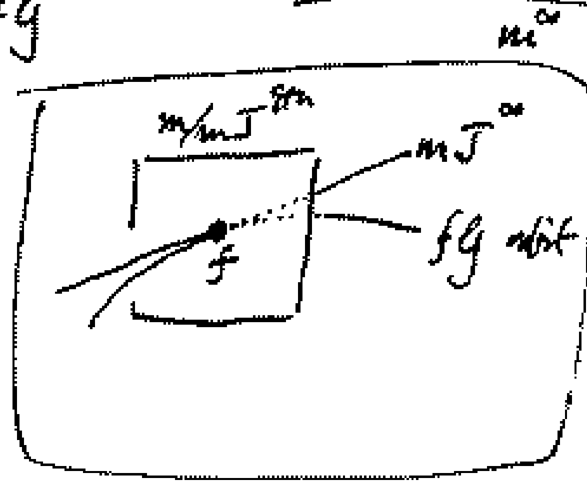
Defn. Given $F: C^q \times R^n \to R$

let $F^+ : C^q \times R^n \to m$ be given by

$$F^+(c,x) = \text{germ at } 0 \text{ of } R^n,0 \longrightarrow R,0$$
$$y \longmapsto F(c,x+y) - F(c,x)$$

Theorem $\left.\begin{array}{l} f \in m^2 \\ f \text{ det} \\ q = \operatorname{codim} f \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \operatorname{codim}(fG \subset m) = q+n \\ F^+ \pitchfork fG \end{array}\right.$

Remark: Links two ideas of codim.



Proof Recall $\underbrace{m \supset J}_{q} \underbrace{\supset mJ}_{n} = T_f(fG)$ Lemma 9

def codim f. Lemma 12

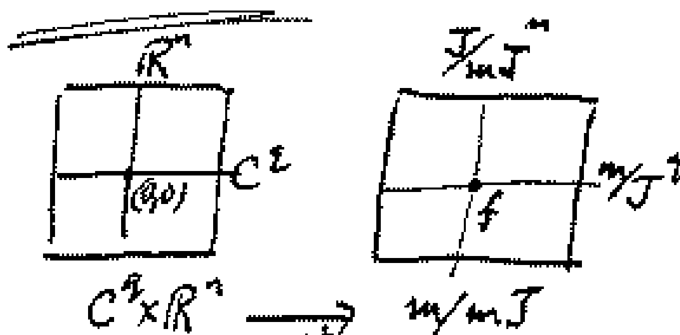
$\therefore \operatorname{codim}(fG \subset m) = \dim \dfrac{T_f m}{T_f(fG)} = \dim \dfrac{m}{mJ} = q+n.$

Recall unfolding $\tilde f : C^q \times R^n \to R$

$\tilde f(c,x) = fx + cx, \quad c = \sum_1^q c_i g_i$

where g_i is basis for m/J

c_i are coords for m/J w.r.t. this basis.



$\tilde f^+(0,0) = \text{germ at } (0,0) \text{ of } y \longmapsto \tilde f(0,y) - \tilde f(0,0) = f.$

$\tilde f^+(c,0) = \text{germ at } (c,0) \text{ of } y \longmapsto \tilde f(c,y) - \tilde f(c,0) = f + c = f + \sum c_i g_i$

$\therefore \dfrac{\partial \tilde f^+}{\partial c_i}(0,0) = g_i$ — span m/J by choice of g_i

$\tilde f^+(0,x) = \text{germ at } (0,x) \text{ of } y \longmapsto \tilde f(0,x+y) - \tilde f(0,x) = \text{germ of } f \text{ at } x$

$\therefore \dfrac{\partial \tilde f^+}{\partial x_i}(0,0) = \dfrac{\partial f}{\partial x_i}$ — span J/mJ by Lemma 12

$\therefore T_f \tilde f^+ = m/J + J/mJ = m/mJ \pitchfork fG.$

Remark 1 The unfolding $\tilde{F}^+$ captures all types of perturbation nearby.

Remark 2 If $F^+ \pitchfork \mathcal{G}$ then F is locally equiv to f at each pt, in sense of Ch 2.
The proof involves the Malgrange Preparation Theorem & is about 10 times as long as proof of Theorems I & II.

DENSITY

$\mathcal{G}$ acts on m, & decomposes m into a union of $\mathcal{G}$-orbits:

i) The open orbit of x_1, codim 0 [non-equil pt]

ii) Morse orbits of $\pm x_1^2 \pm \dots \pm x_n^2$, codim n [min, max, saddle]

iv) fold orbits $\pm x_1^2 \pm \dots \pm x_{n-1}^2 + x_n^3$, codim $1+n$

v) cusp orbits $\pm x_1^2 \pm \dots \pm x_{n-1}^2 \pm x_n^4$, codim $2+n$

} 7 elem cats of codim ≤ 4

vi) all other orbits, of codim $> 4+n$.

Defn: Let $\mathcal{F} = \{F : C^q \times R^n \to R,\ q \leq 4,\ F^+ \pitchfork \text{ all } \mathcal{G} \text{ orbits}\}$.

Theorem $\mathcal{F}$ open dense in $C^\infty(C^q \times R^n)$

$\mathcal{F}$ = the set of locally stables in sense of Ch 2.

If $F \in \mathcal{F}$ then F^+ meets only orbits of codim $\leq 4+n$.

$\therefore$ only singularities of F are elem cats.

The proof is again long, but the reader can now see intuitively how & why the classification works.

For full proof see Zeeman: Catastrophe theory: selected papers 1972–77, Addison Wesley, 1977, chapter 18.

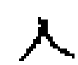
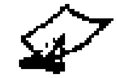
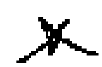
CATASTROPHE THEORY

STANDARD FORMS FOR THE SEVEN ELEMENTARY CATASTROPHES OF CODIM ≤ 4

Control (or parameter) space C^k $(\cong \mathbb{R}^k)$, parameters $a, b, c, \dots$
codimension $k = \dim C \leq 4$.
State space X^n $(\cong \mathbb{R}^n)$, variables $x, y, \dots$

Germ $f_0 : X^n \to \mathbb{R}$

Potential (= unfolding) $f : C^k \times X^n \to \mathbb{R}$

Type	Thom's Name	Arnold's Symbol	n	codim k	Germ	Equivalent more convenient germ f_0	Potential f (= unfolding)	Bifurcation set
Cuspoids	Fold	A_2	1	1	x^3	$\frac{1}{3}x^3$	$\frac{1}{3}x^3 - ax$	•
	Cusp	A_3	1	2	x^4	$\frac{1}{4}x^4$	$\frac{1}{4}x^4 - ax - \frac{1}{2}bx^2$	
	Swallowtail	A_4	1	3	x^5	$\frac{1}{5}x^5$	$\frac{1}{5}x^5 - ax - \frac{1}{2}bx^2 - \frac{1}{3}cx^3$	
	Butterfly	A_5	1	4	x^6	$\frac{1}{6}x^6$	$\frac{1}{6}x^6 - ax - \frac{1}{2}bx^2 - \frac{1}{3}cx^3 - \frac{1}{4}dx^4$	
Umbilics	Hyperbolic	D_4^+	2	3	x^3+y^3	$\frac{1}{3}(x^3+y^3)$	$\frac{1}{3}(x^3+y^3) - ax - by + 2cxy$	
	Elliptic	D_4^-	2	3	x^3-y^3	$\frac{1}{3}x^3 - xy^2$	$\frac{1}{3}x^3 - xy^2 - ax + by + c(x^2+y^2)$	
	Parabolic	D_5	2	4	x^2y+y^4	x^2y+y^4	$x^2y+y^4+ax+by+cx^2+dy^2$	...

Equilibrium manifold $M^k, \subset C^k \times X^n$, given by $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ (or $\nabla f = 0$)

Bifurcation set $B^{k-1}, \subset C^k$, given by $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = H = 0$, $H = $ hessian $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$.

CATASTROPHE THEORY : SHEET 1

CUSP : A_3

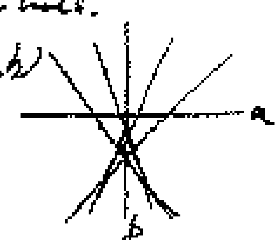
The standard cusp catastrophe is given by $f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, where $f(a,b;x) = \frac{1}{4}x^4 - ax - \frac{1}{2}bx^2$.
The equilibrium surface M is given by $\frac{\partial f}{\partial x} \equiv x^3 - a - bx = 0$, and attractor subsurface M^* by $3x^2 > b$. The fold curve F is given by $3x^2 = b$.

① Prove F is the twisted cubic curve $\{(-2\lambda^3, 3\lambda^2, \lambda);\ \lambda \in \mathbb{R}\}$.

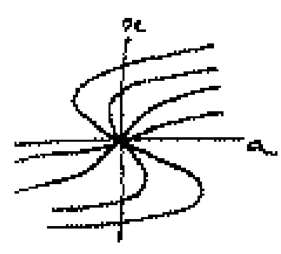
② Fix $b = 3\lambda^2$, where $\lambda > 0$, and increase a from negative to positive. Prove that the jump occurs at $a = 2\lambda^3$ from $x = -\lambda$ to $x = 2\lambda$.

It is important to develop a quantitative feel as well as a qualitative understanding of the cusp. Therefore draw the following sections of M on graph paper.

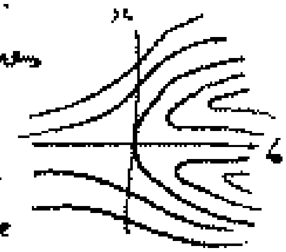
③ Prove that the sections $x =$ constant are straight lines. Prove that the projections of these lines in the (a,b) plane touch the cusp $27a^2 = 4b^3$. Draw these lines in the square $|a| \le 2$, $|b| \le 2$ for the eleven values $x = -1, -0.8, \dots, 1$. Shade the image of M^*.



④ In the (a,x)-plane draw the sections of M $b =$ constant in the square $|a| \le 2$, $|x| \le 2$, for the five values $b = -2, -1, 0, 1, 2$. Prove that these are disjoint except at the origin. Draw the image of F, & shade the image of M^*.



⑤ In the (b,x)-plane draw the sections $a =$ const in the square $|b| \le 2$, $|x| \le 2$ for the five values $a = -2, -1, 0, 1, 2$. Prove these are disjoint (by showing M projects diffeomorphically). Prove that sections $a \ne 0$ have two components, & describe the section $a = 0$. Draw the image of F, & shade the image of M^*.



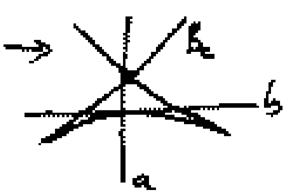
CATASTROPHE THEORY. Sheet 2

SWALLOWTAIL A_4

The standard swallowtail catastrophe $f: C^3\times X^1 \to R$ is given by
$$f(a,b,c;x) = \tfrac{1}{5}x^5 - ax - \tfrac{1}{2}bx^2 - \tfrac{1}{3}cx^3.$$
The equilibrium manifold $M^3, \subset C\times X$, is given by $\frac{\partial f}{\partial x}=0$.
The catastrophe map $\chi: M\to C$ is induced by projection $\pi_1: C\times X\to C$.
The fold surface $F^2, \subset M$, is given by $\partial f/\partial x = \partial^2 f/\partial x^2 = 0$.
The bifurcation set $B^2 = \chi F, \subset C$.
The cusp curve $K^1, \subset F$, is given by $\partial f/\partial x = \partial^2 f/\partial x^2 = \partial^3 f/\partial x^3 = 0$.
The cusped edge $E^1 = \chi K, \subset B$.

(1) Prove that $\pi_2: C\times X \to X$ maps K diffeomorphically onto X.
Deduce that E can be parametrised, using x as a parameter, by
the map $\pi_1(\pi_2|K)^{-1}: X\to E$, sending $x \longmapsto (a,b,c) = (x^4, -8x^3, 6x^2)$.
Deduce that E has a cusp at O, tangent to the c-axis.

(2) Let F^1_c, B^1_c be sections of F, B given by $c =$ constant.
Prove that π_2 maps F_c diffeomorphically onto X.
Deduce that B_c can be parametrised, using x as a parameter, by
the map $\pi_1(\pi_2|F^1_c)^{-1}: X\to B_c$, sending $x \longmapsto (a,b) = (-3x^4+cx^2, 4x^3-2cx)$.
Suppose $c = 6\lambda^2$, $\lambda>0$. Prove that B_c looks like $\to$



with cusps at $(a,b) = (3\lambda^4, \pm 8\lambda^3)$,
and a double point at $(a,b) = (-9\lambda^4, 0)$.

(3) Deduce that B looks like, with a double curve along the half-parabola $4a+c^2=0$, $b=0$, $c>0$.

(4) Prove that the other half of the parabola $(c>0)$ consists of points for which $\exists$ complex x, but no real x, such that $\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = 0$.

(5) Let $\varphi = \frac{\partial f}{\partial x}: C\times X \to R$. Prove $d\varphi \neq 0$ in $C\times X$. Deduce that M is a 3-manifold, & that F is the set of singularities of $\chi: M\to C$.

CATASTROPHE THEORY: SHEET 3

HYPERBOLIC UMBILIC: D_4^+

The standard hyperbolic umbilic catastrophe $f: C^3 \times X^2 \to \mathbb{R}$
is given by $f(a,b,c;x,y) = \frac{x^3+y^3}{3} - ax - by + 2cxy$.
Equilibrium manifold $M^3, \subset C \times X$, is given by $f_x = f_y = 0$, where $f_x = \frac{\partial f}{\partial x}$ etc.
Bifurcation set $B^2, \subset C$, is given by $f_x = f_y = H = 0$, where Hessian $H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$.
Let M_c, B_c, C_c denote sections of M, B, C given by $c =$ constant.

---

(1) Prove that $\forall x, y, c$ $\exists$ unique a, b such that $(a,b,c;x,y) \in M$.
Deduce $\pi_2 : C \times X \to X$ maps M_c diffeomorphically onto X.

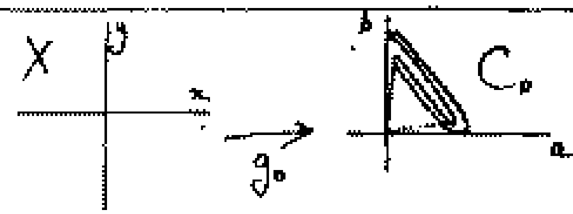
---

(2) Let $g_c = \pi_1 (\pi_2 | M_c)^{-1} : X \to C_c$, $(x,y) \mapsto (a,b)$. Prove g_c is given by $\begin{cases} a = x^2 + 2cy \\ b = y^2 + 2cx \end{cases}$

Deduce that $g_c(\text{Sing } g_c) = B_c$

---

(3) Deduce that g_0 is equivalent to folding X along the axes & mapping into the positive quadrant of C_0.



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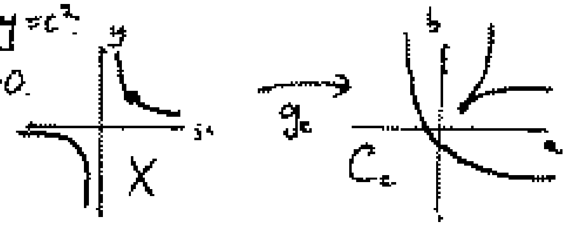
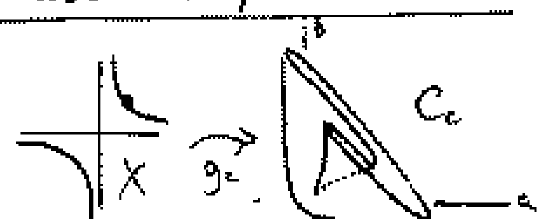
(4) Let $c > 0$. Prove that Sing g_c is given by $xy = c^2$.
Parametrise Sing g_c by $(x,y) = (ct, \frac{c}{t})$, $t \in \mathbb{R} - 0$.
Prove B_c is given by $\begin{cases} a = c^2(t^2 + \frac{2}{t}) \\ b = c^2(2t + \frac{1}{t^2}) \end{cases}$



Calculate $\dot{a}, \dot{b}$, and deduce B_c has a cusp when $t = 1$ at $a = b = 3c^2$.
Deduce B_c has 2 components, a curve and a cusp.

---

(5) Show $g_c X$ covers $\begin{cases} \text{inside the cusp 4 times} \\ \text{between cusp \& curve twice} \end{cases}$
& does not cover outside the curve



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(6) By considering the symmetry $(a,b,c;x,y) \to (a,b,-c;-x,-y)$ show $B_c = B_{-c}$.
Deduce g_c, g_{-c} map opposition components of $xy = c^2$ onto the cusp of B_c.

---

(7) Let M_c^* denote the subset of M_c corresponding to minima of f_c.
Show $\pi_2 M_c^*, \subset X$, is given by $xy > c^2$, $x > 0$, $y > 0$.
Sketch $\pi_2 M_c^*, \subset X$, and $\pi_1 M_c^*, \subset C_c$, for $c < 0$, $c = 0$, & $c > 0$.

---

(8) Sketch $B, \subset C$.

---

(9) Prove B is given by $(ab - 9c^4)^2 = 4(a^2 - 3bc^2)(b^2 - 3ac^2)$

CATASTROPHE THEORY · SHEET 4

ELLIPTIC UMBILIC : D_4^-

The standard elliptic umbilic catastrophe $f : C^3 \times X^2 \longrightarrow R$ is given by

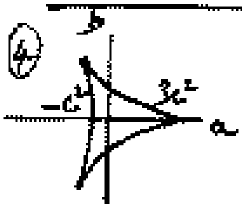
$$f(a,b,c\,;x,y) = \frac{x^3}{3} - xy^2 - ax + by + c(x^2+y^2).$$

Equilibria $M^3 \subset C \times X$, given by $f_x = f_y = 0$
Bifurcation set $B^2 \subset C$, given by $f_x = f_y = H = 0$ $\Big\}$ where $f_x = \frac{\partial f}{\partial x}$ etc., $H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

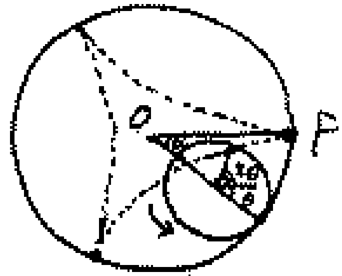
① Let M_c, B_c, C_c be sections of M, B, C given by $c = \text{const}$. Prove that $\forall x, y, c$ $\exists$ unique a, b such that $(a,b,c;x,y) \in M$. Deduce that $\pi_2 : C \times X \to X$ maps M_c diffeomorphically.

② Let $g_c = \pi_1 (\pi_2 | M_c)^{-1} : X \to C_c$. Prove g_c maps $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x^2 - y^2 + 2cx \\ 2xy - 2cy \end{pmatrix}$.
Let $z = x + iy$, $w = a + ib$. Prove $w = g_c z = z^2 + 2c\bar{z}$

③ Let Γ be the circle $|z| = |c|$. Prove $\Gamma = \text{Sing}\, g_c$, given by $\begin{vmatrix} \partial a/\partial x & \partial a/\partial y \\ \partial b/\partial x & \partial b/\partial y \end{vmatrix} = 0$.

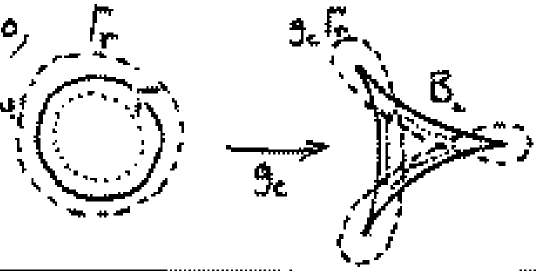
④ 

Prove $B_c = g_c \Gamma$ = curve in the (a,b)-plane given by
$$w = c^2 (e^{2i\theta} + 2e^{-i\theta}), \quad 0 \le \theta < 2\pi.$$
Deduce B_c is the triangular hypocycloid given by the locus of a point P on a circle of radius c^2 rolling inside the circle centre O radius $3c^2$.

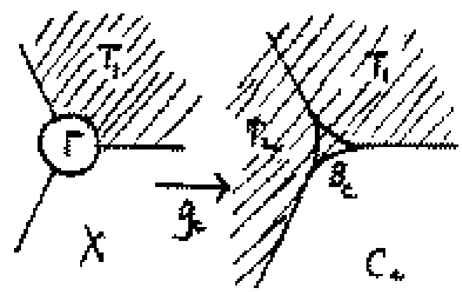


Prove, by putting $\frac{dw}{d\theta} = 0$, that the cusp points are $3c^2 \times$ (cube roots of 1).

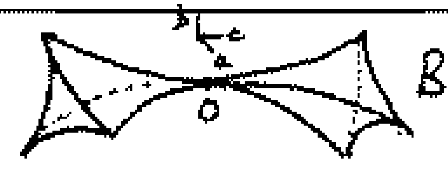
⑤ Let Γ_r be the circle $|z| = r$. Show that, for $c > 0$,
$r < c \Rightarrow g_c \Gamma_r$ is a curve inside B_c without self-intersections,
$r > c \Rightarrow g_c \Gamma_r$ has three self-intersections, as shown.



⑥ For $j = 1,2,3$ let T_j = the trecant $\frac{2\pi}{3}(j-1) \le \theta \le \frac{2\pi}{3} j$.
Prove that if $c > 0$ then g_c maps
$$T_1 \cap (\text{outside } \Gamma) \xrightarrow[\text{onto}]{\text{homeo}} T_1 \cup T_2 \cup (\text{inside } B_c).$$
Deduce that $g_c X$ covers $\Big\{$ the outside of B_c twice, and the inside four times.



⑦ Deduce from ④ that B looks like this $\longrightarrow$
Prove B meets $b = 0$ in two parabolas,
$a = 3c^2$ and $a = -c^2$.



⑧ Let M^* = the minima of f = the subset of M where the Hessian H is positive defi[nite].
Prove $\pi_1 M^* = (\text{inside } B) \cap (c > 0)$

CATASTROPHE THEORY: Sheet 5

BUTTERFLY CATASTROPHE: A_5

The standard butterfly $f: C^4 \times X^1 \to R$ is given by

$$f(a,b,c,d;x) = \tfrac{1}{6}x^6 - ax - \tfrac{1}{2}bx^2 - \tfrac{1}{3}cx^3 - \tfrac{1}{4}dx^4$$

Equilibrium manifold $M^4 \subset C\times X$, given by $\partial f/\partial x = 0$.

Bifurcation set $B^3 = \pi_1 F^3 \subset C$, where F given by $\partial f/\partial x = \partial^2 f/\partial x^2 = 0$.

Let M_e, F_e, B_e be sections given by $e=(c,d)=$ constant.

(1) Prove $g = \pi_1(\pi_2|F_e)^{-1} : X \to B_e$ is given by $\begin{cases} a = -4x^5 + cx^2 + 2dx^3 \\ b = 5x^4 - 2cx - 3dx^2 \end{cases}$

Using x as time, calculate $\dot a, \dot b, \frac{da}{db}$, & explain what these mean geometrically.

(2) Prove that if $c=0, d<0$ then B_e has only one cusp, at the origin. Draw B_e, M_e.

(3) Prove that if $c>0, d<0$ then B_e has only one cusp, and this lies in the quadrant $a>0$, $b<0$. [Hint: where does $\frac{\partial^3 f}{\partial x^3}$ cross x axis?] Prove B_e touches the b-axis at the origin. [Hint: small x] Draw B_e, M_e.

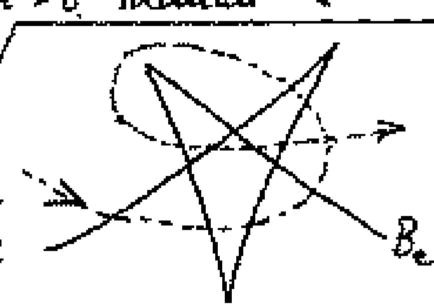
(4) Prove that if $c=0$, $d = \frac{(10)\lambda^2}{3}$, $\lambda>0$, then B_e has 3 cusps at $(0,0)$, $(\pm \frac{8\lambda^5}{3}, -5\lambda^4)$ and 3 double points, one of which is at $(0, -\frac{25}{9}\lambda^4)$. Verify the signs of $\dot a, \dot b$ in the ranges $x>0$, $0<x<\lambda$, $-\lambda<x<0$, $x<-\lambda$. Draw B_e, M_e.

(5) Prove that B_e has a swallowtail point if $5c^2 = 2d^3$, & that if $e>0$ then this lies in the positive quadrant $a>0, b>0$.

(6) In question (4) draw 4 sections of M in the (a,x)-plane for 4 values of b, one in each of the 4 intervals separated by $0, -\frac{25}{9}\lambda^4, -5\lambda^4$. Indicate the stable parts, & the catastrophes.

(7) Let $a=c=0$, & draw 2 sections of M in the (b,x)-plane for 2 values of d, $d<0$ and $d>0$. Indicate the stable parts & the catastrophes.

(8) Indicate where on the dotted path the catastrophic jumps will occur. Invent a notation for indicating all possible jumps.



CATASTROPHE THEORY: Sheet 6

EVOLUTES & BUOYANCY

① Let X be the ellipse $\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = 1$, where $\alpha > \beta > 0$.
Prove that the centre of curvature of X at $(\alpha\cos\theta, \beta\sin\theta)$ is

$$\left(\left(\alpha - \frac{\beta^2}{\alpha}\right)\cos^3\theta,\ \left(\beta - \frac{\alpha^2}{\beta}\right)\sin^3\theta\right).$$

Deduce that the evolute E of X is given by

$$(\alpha x)^{\frac{2}{3}} + (\beta y)^{\frac{2}{3}} = (\alpha^2 - \beta^2)^{\frac{2}{3}}$$

Deduce that the radius of curvature of X at $(\alpha, 0)$ is $\frac{\beta^2}{\alpha}$.
Deduce that E has four cusps.
Show E lies inside $X \iff \alpha < \sqrt{2}\beta$.
Sketch X, E in the two cases (i) $\alpha = 2, \beta = 1$
(ii) $\alpha = 4, \beta = 3$.

---

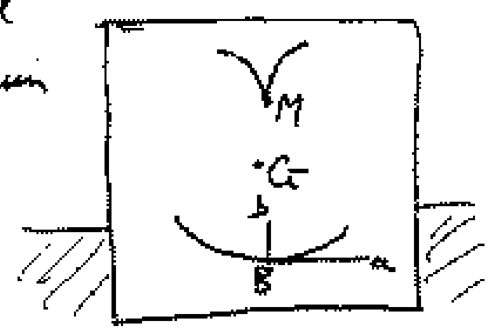
② Show that the evolute of the rectangular hyperbola $x^2 - y^2 = a^2$ is $x^{2/3} - y^{2/3} = (2a)^{2/3}$. [Hint: use parameters $(a\cosh\theta, a\sinh\theta)$]

Deduce that the radius of curvature at $(a, 0)$ is a.

---

③ A log of square cross-section, side 1, and density δ floats in stable equilibrium in a liquid of density 1, as shown.
Taking axes at the centre of buoyancy B, prove the buoyancy locus is $b = 6\delta a^2$ locally. Deduce that the metacentre $M = \left(0, \frac{1}{12\delta}\right)$. Deduce that stability implies

$$\text{either } 0 < \delta < \frac{3-\sqrt{3}}{6} \quad \text{or} \quad \frac{3+\sqrt{3}}{6} < \delta < 1.$$



What happens if $\frac{3-\sqrt{3}}{6} < \delta < \frac{3+\sqrt{3}}{6}$?

CATASTROPHE THEORY : Sheet 7

(1) Define $a : \mathbb{R} \to \mathbb{R}$ by $a(r) = \begin{cases} r(1-e^{-1/r}), & r > 0 \\ r, & r \le 0. \end{cases}$

Prove a C^∞, monotonic increasing, and $a \to 1$ as $r \to \infty$.

Define $b : \mathbb{R} \to \mathbb{R}$ by $b(r) = \varepsilon\left(1 + a\left(\frac{r}{\varepsilon} - 1\right)\right)$, where $\varepsilon > 0$.

Prove b maps $[0, \infty)$ diffeomorphically onto $[0, 2\varepsilon)$, keeping $[0, \varepsilon]$ pointwise fixed.

Define $c : \mathbb{R}^n \to \mathbb{R}^n$ by $c(x) = \frac{x}{|x|}\, b(|x|)$.

Prove c maps $\mathbb{R}^n$ diffeomorphically onto $B_{2\varepsilon}$, keeping B_ε pointwise fixed.

(2) Let E = ring of germs at 0 of C^∞-functions $\mathbb{R}^n \to \mathbb{R}$, and m = max. ideal. Prove by induction, or otherwise, that $\dim(E/m^{k+1}) = \frac{(n+k)!}{n!\,k!}$

(3) Prove the determinacy & codimension of the following germs, & unfold them.

Germ	Determinacy	Codimension
Cuspoid : $x^2 + y^k$, $k \geq 2$	k	$k-2$
Umbilic : $x^2y + y^k$, $k \geq 3$.	k	k
Hyperbolic umbilic : $x^3 + y^3$	3	3
E_6 : $x^3 + y^4$	4	5

(4) State & prove the Splitting Lemma.

(5) Apply the Splitting Lemma to $x^2 + 2xy^2$, and deduce that it is a dual cusp. Calculate its determinacy, codimension & unfolding.

(6) Find a germ with the same 3-jet as, but not equivalent to, $x^2 + 2xy^2$.

(7) Prove (i) $x^2y + y^4 \sim -x^2y + y^4$ (ii) $x^2y + y^4 \not\sim x^2y - y^4$

(8) Prove $m^{k+1} \subset m^2 J \implies f$ k-determinate.

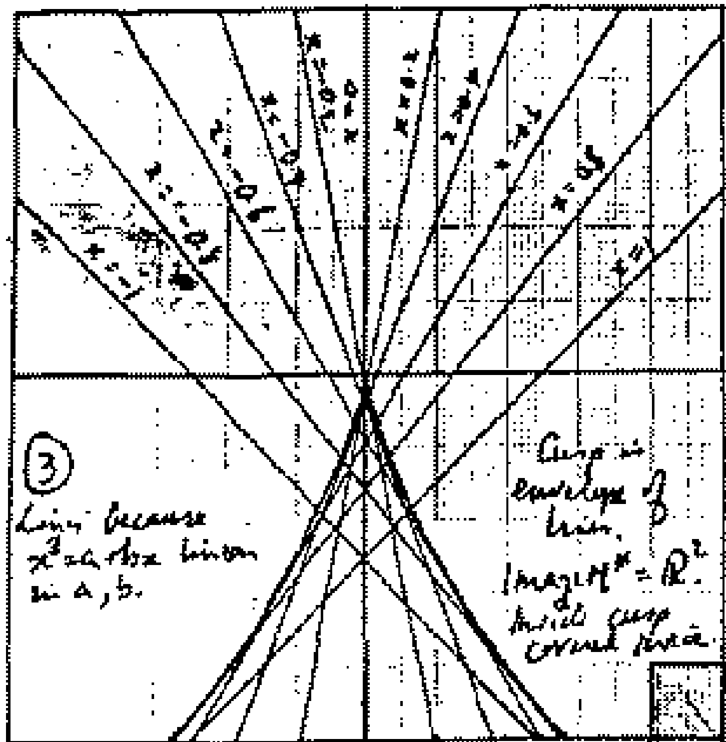
(9) Use (8) to prove the determinacy of the following germs. Also verify their codimensions, and unfold them.

Germ	Determinacy	Codimension
Double cusp : $x^4 + y^4$	4	8
Double swallowtail : $x^5 + y^5$	6	15
Treble fold : $x^3 + y^3 + z^3$	3	7

(10) Explain the apparent paradox that the jets $j^k f$ and $j^{k-1} f$ of a germ f are invariant (independent of coordinates), but their difference, the kth term of the Taylor Series f_k, is <u>not</u> invariant. Give an example.

CATASTROPHE THEORY

Solutions Sheet 1.



① $\forall \lambda \in \mathbb{R}$, let $x = \lambda$

$\therefore b = 3x^2 = 3\lambda^2$

$\therefore a = x^3 - bx = \lambda^3 - 3\lambda^3 = \ldots$

② Catastrophe occurs when path meets fold curve F $\begin{cases} a = 2\lambda^3 \\ x = -\lambda \end{cases}$

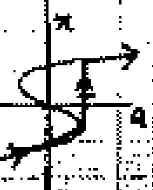
Line $a = 2\lambda^3$ meets M when $x^3 = 2\lambda^3 + 3\lambda^2 x$

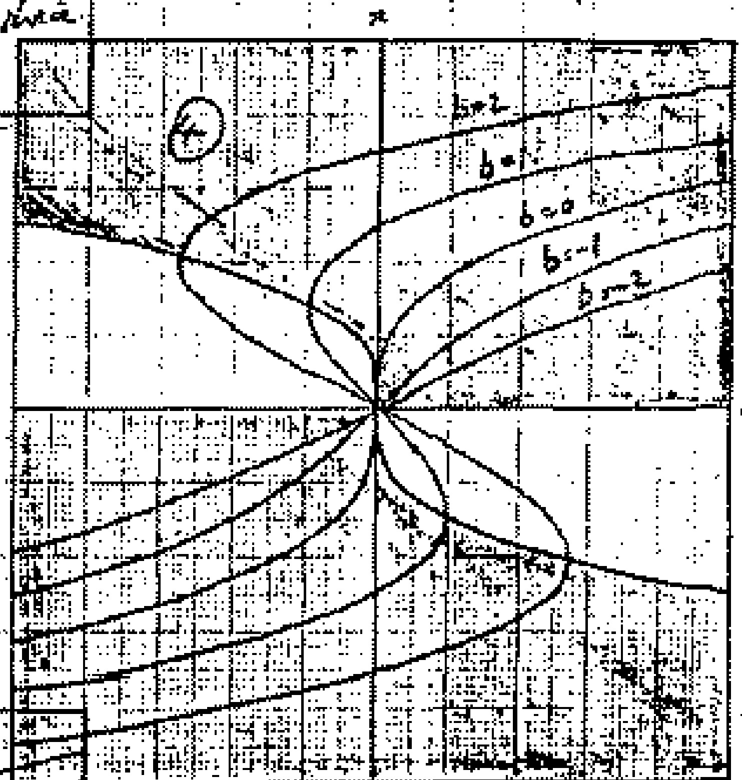
$\therefore x^3 - 3\lambda^2 x - 2\lambda^3 = 0$

$\therefore (x+\lambda)(x+\lambda)(x-2\lambda) = 0$

$\therefore x = -\lambda, -\lambda, 2\lambda$

$\therefore$ Jump from $x = -\lambda$ to $x = 2\lambda$





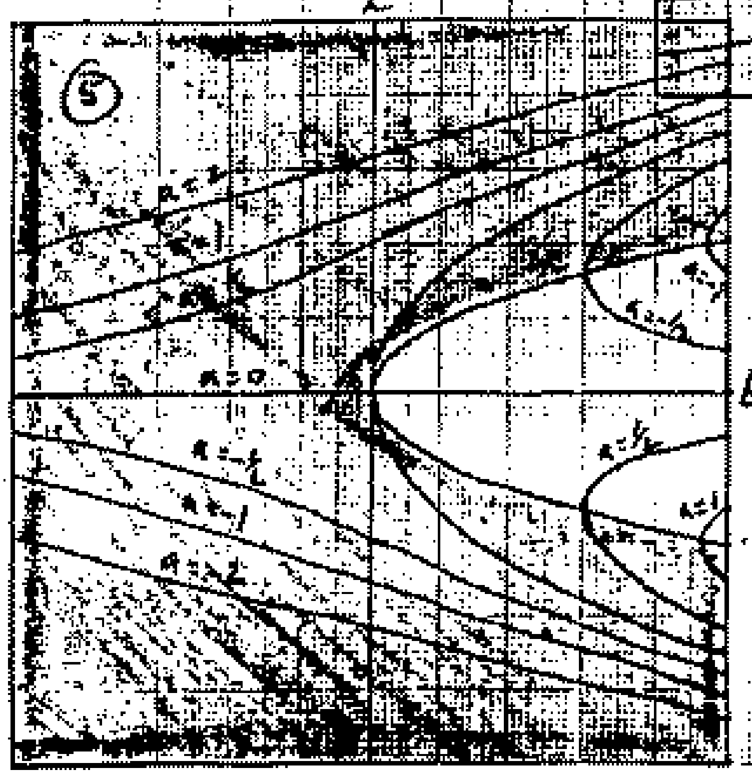
④ Suppose $\begin{cases} x^3 = a + bx \\ x^3 = a + b'x \end{cases}$, $b \neq b'$

$\therefore 0 = (b - b')x$

$\therefore x = 0$, $\therefore a = 0$

$\therefore$ two curves disjoint except at O.

Image M^* bounded by $\frac{1}{2}$ in F & a-axis



⑤ $\forall (a,b) \in \mathbb{R}^2$, $\exists$ unique $a = x^3 - bx$ unique point of M

$\therefore M \to \mathbb{R}^2$ diffeo.

$\forall a \neq 0$, $b = x^2 - a/x \to \infty$ as $x \to 0$

$\therefore$ has 2 components, $x \gtrless 0$.

If $a = 0$, $x(x^2 - b) = 0$

$\therefore$ parabola & b-axis.

Im M^* bounded by $\frac{1}{2}$ in F.

CATASTROPHE THEORY : Solution Sheet 2.

① K is given by $\begin{cases} \partial f/\partial x = x^4 - a - bx - cx^2 = 0 \\ \partial^2 f/\partial x^2 = 4x^3 - b - 2cx = 0 \\ \partial^3 f/\partial x^3 = 12x^2 - 2c = 0 \end{cases}$

$\therefore\ c = 6x^2$

$\therefore\ b = 4x^3 - 2cx = 4x^3 - 2(6x^2)x = -8x^3$

$\therefore\ a = x^4 - bx - cx^2 = x^4 - (-8x^3)x - (6x^2)x^2 = x^4.$

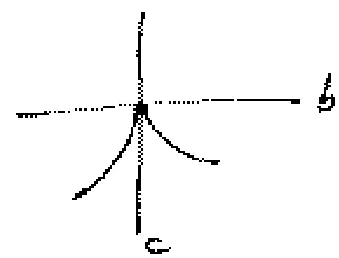
$\therefore\ \forall x \in X,\ \exists!\ (a,b,c;x) = (x^4, -8x^3, 6x^2; x) \in K$ that projects to x.

$\left.\begin{array}{l} \therefore\ K \xrightarrow{\pi_2|K} X \text{ is bijective} \\ \text{It is differentiable because it is a projection} \\ \text{The inverse is differentiable because it is polynomial} \end{array}\right\} \therefore$ diffeo.

$\therefore\ \pi_1(\pi_2|K)^{-1}: X \to E$ maps $x \longmapsto (a,b,c) = (x^4, -8x^3, 6x^2).$

For small x, $(a,b,c) \sim (0, -8x^3, 6x^2)$

which is a cusp.



Near cusp point $(a,b,c) \sim (0, 0, 6x^2)$,

which is tangent to the c-axis.

---

② Fix c. Then F_c is given by $\begin{cases} x^4 - a - bx - cx^2 = 0 \\ 4x^3 - b - 2cx = 0 \end{cases}$

$\therefore\ b = 4x^3 - 2cx$

$\therefore\ a = x^4 - (4x^3 - 2cx)x - cx^2 = -3x^4 + cx^2.$

$\therefore\ \forall x \in X,\ \exists!\ (a,b;x) = (-3x^4 + cx^2, 4x^3 - 2cx; x) \in F_c$ that projects to x

$\left.\begin{array}{l} \therefore\ F_c \xrightarrow{\pi_2|F_c} X \text{ is bijective} \\ \text{Diff}^{ble} \text{ because it is a projection} \\ \text{Inverse diff}^{ble} \text{ because polynomial} \end{array}\right\} \therefore$ diffeo.

$\therefore\ \pi_1(\pi_2|F_c)^{-1}: X \to B$, maps $x \longmapsto (a,b) = (-3x^4 + cx^2, 4x^3 - 2$

② (cont). Suppose $c = 6\lambda^2$, $\lambda > 0$.

$$\therefore \begin{cases} a = -3x^4 + 6\lambda^2 x^2 \\ b = 4x^3 - 12\lambda^2 x \end{cases}$$

Use x as a time-parameter to run along B_c:

$$\left.\begin{cases} \dot a = \frac{\partial a}{\partial x} = -12x^3 + 12\lambda^2 x = -12x(x^2-\lambda^2) \\ \dot b = \frac{\partial b}{\partial x} = 12x^2 - 12\lambda^2 = 12(x^2-\lambda^2) \end{cases}\right\} \therefore \frac{\partial a}{\partial b} = \frac{\dot a}{\dot b} = \text{[illegible]}$$

When $x = \pm\lambda$ then $\dot a = \dot b = 0$.

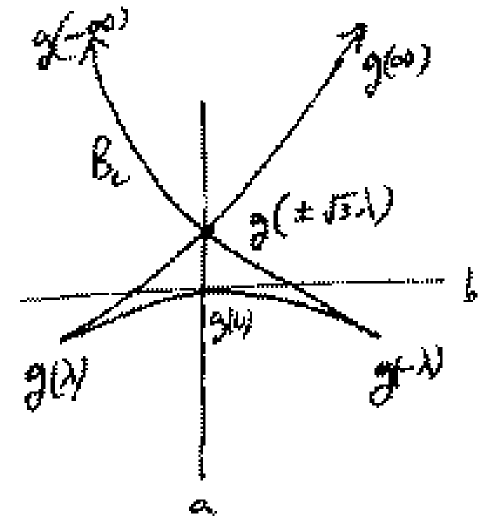
$\therefore$ there are cusps at $(a,b) = (-3\lambda^4 + 6\lambda^2\lambda^2, \pm 4\lambda^3 \mp 12\lambda^3)$

$= (3\lambda^4, \mp 8\lambda^3)$

When $x \neq \pm\lambda$ then $(\dot a, \dot b) \neq (0,0)$ & so these are the only cusps

Then

	$\dot a$	$\dot b$	Slope $\frac{\partial a}{\partial b}$
$x < -\lambda$	+	+	+
$x = -\lambda$	0	0	+
$-\lambda < x < 0$	−	−	+
$x = 0$	0	−	0
$0 < x < \lambda$	+	−	−
$x = \lambda$	0	0	−
$x > \lambda$	−	+	−



$g(0) = (0,0)$

For small x, $g(x) \sim (6\lambda^2 x^2, -12\lambda^2 x)$ = parabola, axis the a-axis.

As $x \to \infty$ $g(x) \sim (-3x^4, 4x^3) \to (-\infty, \infty)$

$x \to -\infty$ $g(x) \to (-\infty, -\infty)$

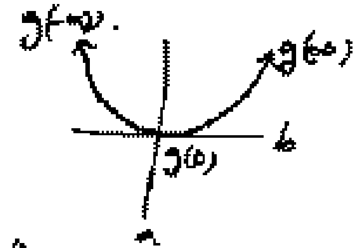
Double point when $b = 0$, $x \neq 0$. $\therefore 4x^3 - 12\lambda^2 x = 0$. $\therefore 4x(x^2 - 3\lambda^2) = 0$

$\therefore x = \pm\sqrt{3}\lambda$.

$\therefore a = -3(\sqrt{3}\lambda)^4 + 6\lambda^2(\sqrt{3}\lambda)^2 = -27\lambda^4 + 18\lambda^4 = -9\lambda^4$.

③ If $c > 0$ there are no cusps because $b^2 > 0$

$\therefore$ B_c looks like



Hence B looks like

cusped edge by question ①

Double curve given by $c = 6\lambda^2 > 0$

$$a = -9\lambda^4, \quad b = 0.$$

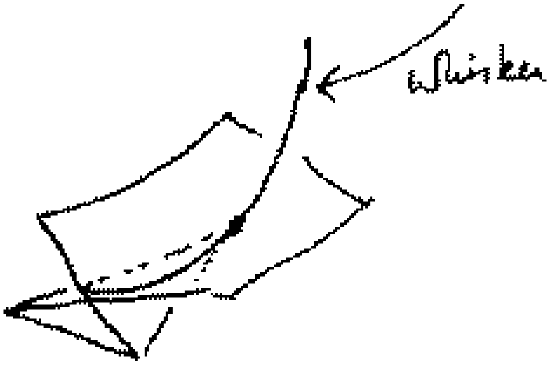
$$\therefore \quad c^2 = 36\lambda^4 = -4a.$$

$$\therefore 4a + c^2 = 0.$$

$\therefore$ half parabola $\begin{cases} 4a + c^2 = 0 \\ b = 0 \\ c > 0 \end{cases}$

---

④ The other half parabola is parametrised $\quad a = -9\lambda^4$, $b = 0$, $c = -6\lambda^2$



$$\therefore \; x^4 - a - bx - cx^2 = x^4 + 9\lambda^4 + 6\lambda^2 x^2 = (x^2 + 3\lambda^2)^2$$

$$4x^3 - b - 2cx = 4x^3 + 12\lambda^2 x = 4x(x^2 + 3\lambda^2)$$

Both vanish $\iff x^2 + 3\lambda^2 = 0$
$\iff x = \pm\sqrt{3}\, i\lambda$, complex

(5) Let p denote a point $p=(a,b,c,x)\in C\times X$.

Let $\varphi = x^4 - a - bx - cx^2 : C\times X \longrightarrow \mathbb{R}$.

$$\therefore d\varphi = \left(\frac{\partial\varphi}{\partial a}, \frac{\partial\varphi}{\partial b}, \frac{\partial\varphi}{\partial c}, \frac{\partial\varphi}{\partial x}\right)$$

$$= \left(-1, -x, -x^2, 4x^3 - b - 2cx\right)$$

$$\neq 0, \quad \forall p \in C\times X.$$

M is given by $\varphi = 0$.

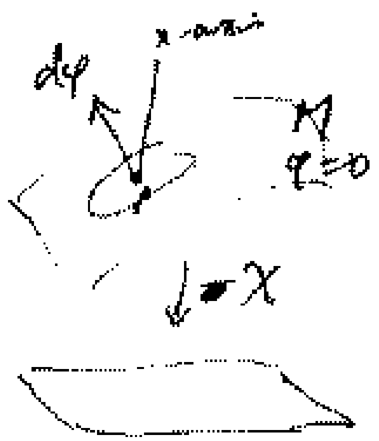
$\therefore$ by the implicit function theorem M is a manifold of codimension 1, i.e. a 3-manifold.

Furthermore $d\varphi$ is the normal to M at p, $\forall p$.

Now consider $\chi : M \to C$.

p is a regularity of χ $\Leftrightarrow$ M transversal to x-axis at p

$\Leftrightarrow$ normal to M at p $\not\perp$ x-axis



$\Leftrightarrow$ x-coordinate of $d\varphi \neq 0$

$\Leftrightarrow$ $\frac{\partial\varphi}{\partial x} \neq 0$

$\Leftrightarrow$ $\frac{\partial f}{\partial x} \neq 0$.

$\Leftrightarrow$ $p \notin F$.

$\therefore$ p is a singularity $\Leftrightarrow$ $p \in F$.

$\therefore$ Singular set of χ $= F$.

CATASTROPHE THEORY : Solution Sheet 3

Hyperbolic umbilic.

M given by $\begin{cases} f_x = x^2 - a + 2cy = 0 \\ f_y = y^2 - b + 2cx = 0 \end{cases}$

$\forall x, y, c$ $\exists$ unique a, b given by $\begin{cases} a = x^2 + 2cy \\ b = y^2 + 2cx \end{cases}$ such that $(a, b, c; x, y) \in M$.

Hence $\pi_2 | M_c : M_c \to X$ is bijective
differentiable because induced by projection
inverse differentiable because polynomial $\Big\}$ hence diffeo.

---

(2) Formula for g_c from question (1).

Since $\pi_1|M_c$ & g_c are related by the diffeo $\pi_2|M_c$, their singularities are diffeomorphic.

$$\begin{tikzcd} M_c \arrow[rr, "\pi_2|M_c", "\cong"'] \arrow[dr, "\pi_1|M_c"'] & & X \arrow[dl, "g_c"] \\ & C_c & \end{tikzcd}$$

$\therefore$ $g_c(\text{Sing } g_c) = \pi_1(\text{Sing } \pi_1|M_c) = B_c$

---

(3) Case $c = 0$ $\quad \begin{rcases} a = x^2 \\ b = y^2 \end{rcases}$ is equivalent to folding X along its axes

---

(4) Case $c > 0$ The Jacobian of g_c is $J = \begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2c \\ 2c & 2y \end{vmatrix} = 4(xy - c^2)$

Sing g_c are given by $J = 0$. $\therefore$ $\underline{xy = c^2}$

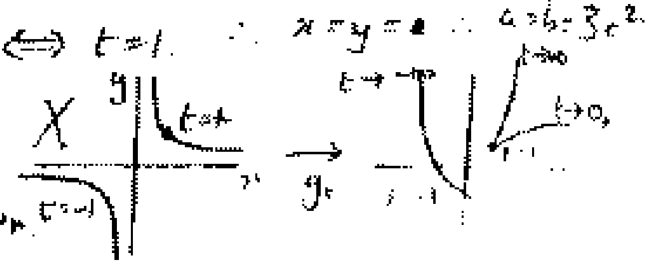
Parametrise by $(x, y) = (ct, \frac{c}{t})$, $t \in \mathbb{R} - 0$.

$\therefore \begin{cases} a = x^2 + 2cy = (ct)^2 + 2c\frac{c}{t} = c^2(t^2 + \frac{2}{t}) \\ b = y^2 + 2cx = (\frac{c}{t})^2 + 2c.ct = c^2(2t + \frac{1}{t^2}) \end{cases}$

$\therefore \begin{cases} \dot{a} = c^2(2t - \frac{2}{t^2}) = \frac{2c^2}{t^2}(t^3 - 1) \\ \dot{b} = c^2(2 - \frac{2}{t^3}) = \frac{2c^2}{t^3}(t^3 - 1) \end{cases}$ $\quad \therefore$ Slope $= \frac{\dot{b}}{\dot{a}} = t$.

Hence $\dot{a} = \dot{b} = 0 \iff t^3 = 1 \iff t = 1$. $\therefore x = y = c$ $\therefore a = b = 3c^2$.

If $t > 0$, slope $= t$ ensures cusp lies in positive quadrant.
If $t = -1$, $a = b = -c^2$, and if $t < 0$ slope ensures monotonic in shape.



(5) Let $N(a,b)$ = number of times $f_c X$ covers $(a,b) \in C_c$.

Then $N : C_c \to \mathbb{Z}$ is $\begin{cases} \text{discontinuous on } B_c \\ \text{continuous on } C_c - B_c \end{cases}$

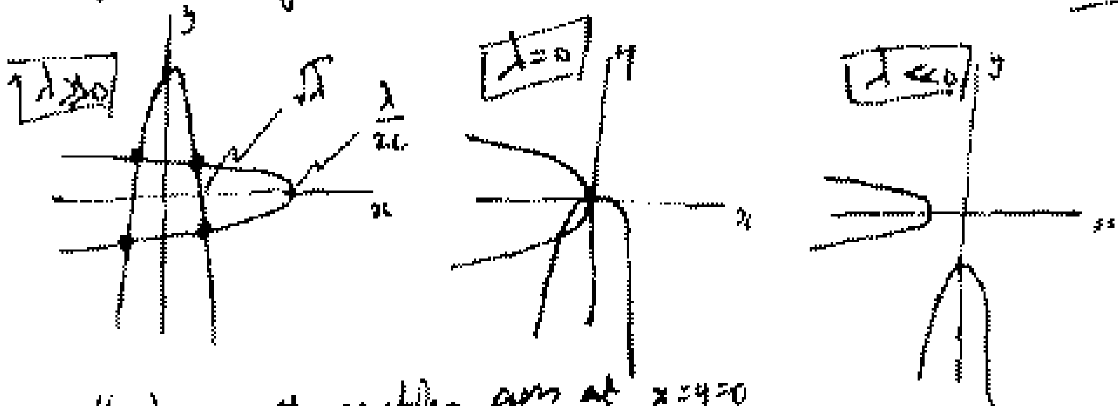
$\therefore$ N is constant on each component of $C_c - B_c$.

$\therefore$ it suffices to calculate N at one point in each component.

Choose a point (λ, λ) on the diagonal.

Then $N(\lambda,\lambda)$ is the number of solutions (x,y) of $\begin{cases} x^2 + 2cy = \lambda \\ y^2 + 2cx = \lambda \end{cases}$

If λ is large then parabolas cross 4 times, since $\frac{\lambda}{2c} > \sqrt{\lambda}$. $\therefore N = 4$



If $\lambda = 0$ the parabolas cross at $x = y = 0$
& $x = y = -2c$.

$\therefore N = 2$

If $\lambda \ll 0$ the parabolas miss. $\therefore N = 0$

(6) Involution $\theta : C \times X \to C \times X$ given $\theta(a,b,c,x,y) = (a,b,-c,-x,-y)$ maps $f \to f$ & hence preserves M, since M given by $f_x = f_y = 0$.

$\therefore$ θ preserves $B = \pi_1(\text{Sing } \pi_1 | M)$ $\therefore \theta B_c = B_{-c}$

$\therefore$ in the (a,b)-plane $B_c = B_{-c}$.

The diagram
$$\begin{array}{ccccc} X & \xrightarrow{f_c} & C_c & \cong & \mathbb{R}^2 \\ \downarrow \theta|X_c = -1 & & \cong \downarrow \theta|C_c & & \downarrow 1 \\ X & \xrightarrow{f_{-c}} & C_{-c} & \cong & \mathbb{R}^2 \end{array}$$
is commutative

$\theta|X = -1$, switches components of $xy = c^2$

$\therefore$ 1 preserves .. .. B_c

$\therefore$ -1, maps opposite components of $xy = c^2$ onto the cusp of B_c.

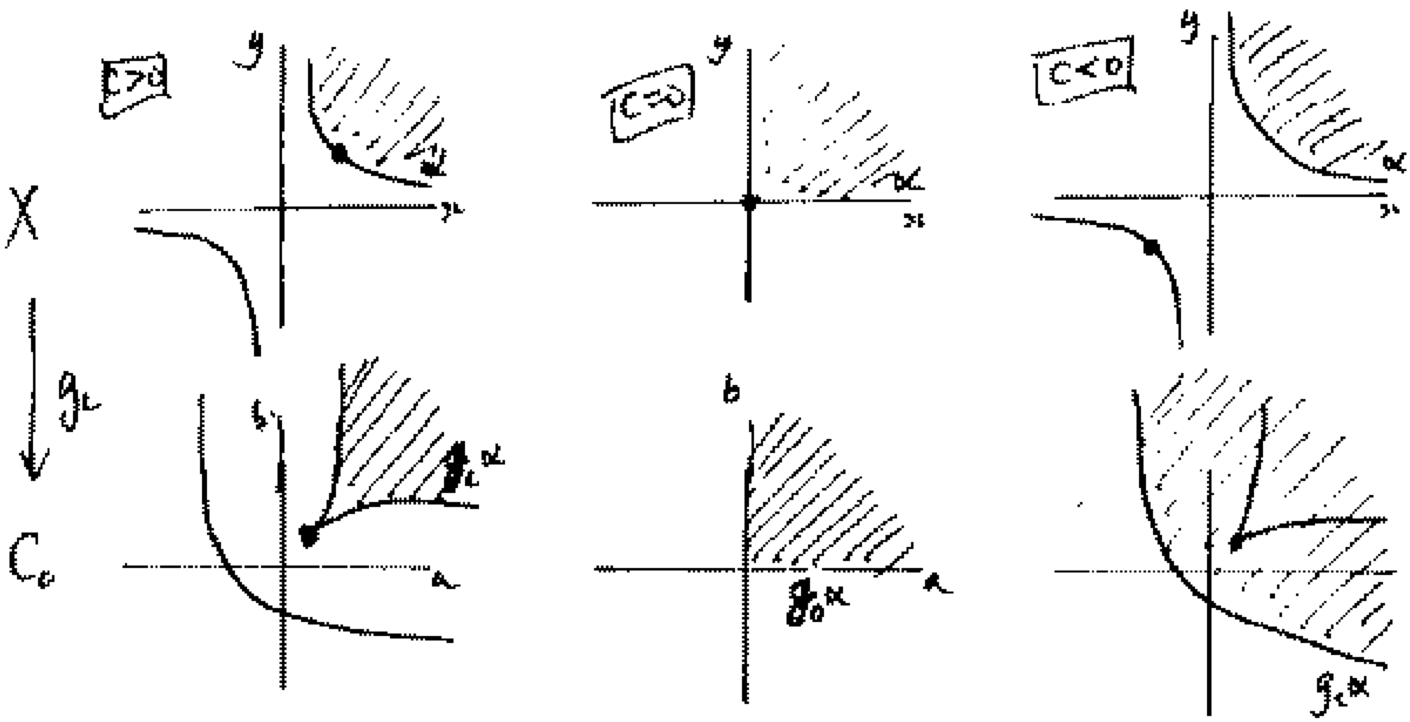
(7) M^* is the subset of M given by positive definite Hessian H,

where $H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2x & 2c \\ 2c & 2y \end{vmatrix}$

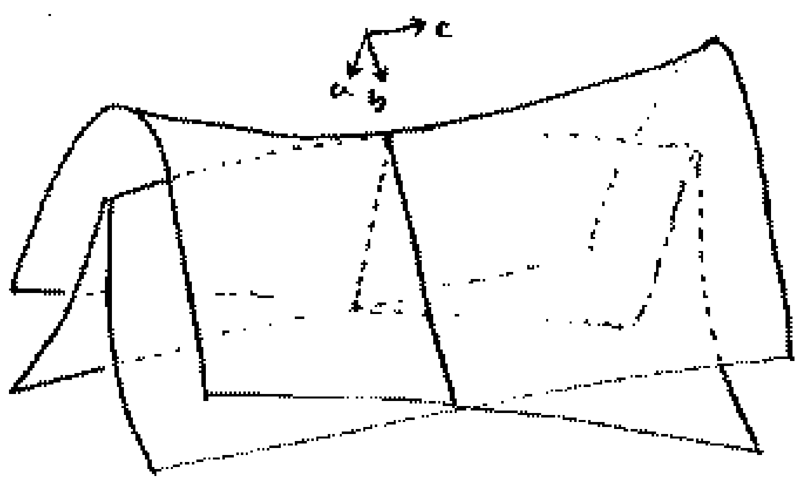
$\therefore xy > c^2$ & $x > 0$ (& hence $y > 0$)

$\therefore \pi_2 M_c^*$, $\subset X$, is the component of $X - \mathrm{Sing}\, f_c$ in the positive quadrant.

By (5) & (6) $\pi_1 M_c^* = f_2(\pi_2 M^*)$ is the open region of C_c bounded by $f_c\alpha$, shown shaded.



(8)



equation for B is obtained by eliminating x, y from

$$x^2 - a + 2cy = 0 \quad (1)$$
$$y^2 - b + 2cx = 0 \quad (2)$$
$$xy = c^2 \quad (3)$$

(1)y: $\quad x^2y - ay + 2cy^2 = 0$

Subst (3) $\quad c^2x - ay + 2cy^2 = 0.$

2c(2) $\quad 2cy^2 - 2bx + 4c^2x = 0.$

Subtract $\quad -3c^2x - ay + 2bc = 0$

Rewrite: $\quad 3c^2x + ay = 2bc \quad (4)$

Symmetry: $\quad 3c^2y + bx = 2ac \quad (5)$

$3c^2$(4) $\quad 9c^4x + 3ac^2y = 6bc^3$

a(5) $\quad 3ac^2y + abx = 2a^2c$

Subtract $\quad (ab - 9c^4)x = 2c(a^2 - 3bc^2)$

Symmetry: $\quad (ab - 9c^4)y = 2c(b^2 - 3ac^2)$

Multiply: $\quad (ab-9c^4)^2xy = 4c^2(a^2-3bc^2)(b^2-3ac^2)$

Subst (3): $\quad (ab-9c^4)^2 = 4(a^2-3bc^2)(b^2-3ac^2)$

CATASTROPHE THEORY : SOLUTION SHEET 4

Elliptic umbilic.

(1) M is given by $\begin{cases} f_x = x^2 - y^2 - a + 2cx = 0 \\ f_y = -2xy + b + 2cy = 0 \end{cases}$

$\therefore\ \forall x,y,c\ \exists$ unique a,b given by $\begin{cases} a = x^2 - y^2 + 2cx \\ b = 2xy - 2cy \end{cases}$

$\therefore\ \pi_c|M_c : M_c \xrightarrow{\text{bijective}} X.$

Differentiable, since it is projection.
Inverse also, since polynomial $\Big\}$ $\therefore$ diffeo

(2) χ_c given by question (1) $\quad\therefore\ w = a + ib$

$$= (x^2 - y^2 + 2cx) + i(2xy - 2cy)$$
$$= (x+iy)^2 + 2c(x - iy)$$
$$= z^2 + 2c\bar{z}$$

(3) Singularities of χ_c given by:

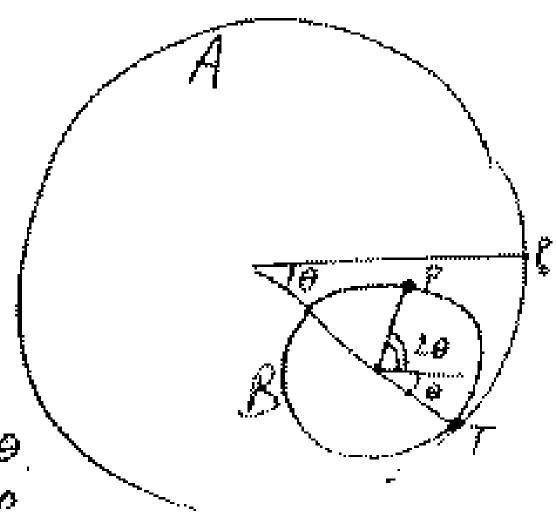
Jacobian, $J(\chi_c) = \begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x+2c & -2y \\ 2y & 2x-2c \end{vmatrix} = 4(x^2 - c^2 + y^2) = 0$

$\therefore\ x^2 + y^2 = c^2$

$\therefore\ |z| = |c|. \qquad \therefore$ Sing $\chi_c = \Gamma$.

(4) $z \in \Gamma \Rightarrow z = ce^{i\theta},\ 0 \le \theta < 2\pi$

$\therefore\ w = f_c z = z^2 + 2c\bar{z} = c^2 e^{2i\theta} + 2c.ce^{-i\theta}$
$$= c^2(e^{2i\theta} + 2e^{-i\theta})$$



Let A = circle centre O, radius $3c^2$.
$S = 2c^2 e^{-i\theta}$
B = circle centre S, radius c^2
$\therefore\ B$ touches A at $3c^2 e^{-i\theta}$ (= T say)
Let TP_0, TP = arcs in A, B of angle θ, 3θ.
$\therefore$ length $TP_0 = 3c^2 \times \theta = c^2 \times 3\theta$ = length TP
$\therefore$ locus of P obtained from P_0 by rolling B inside

Cusps are the critical points of $g_c\Gamma$

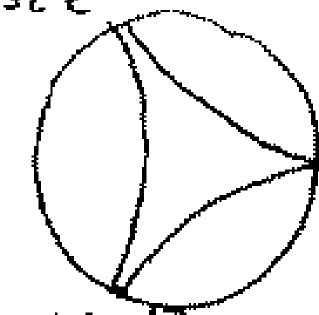
$\therefore c^2(2ie^{2i\theta} - 2ie^{-i\theta}) = 0$

$\therefore e^{3i\theta} = 1$

$\therefore 3\theta = 0 \mod 2\pi$

$\therefore \theta = 0, \pm \frac{2\pi}{3}$

$\therefore w = 3c^2, \ 3c^2 e^{\mp \frac{2\pi}{3}i}$



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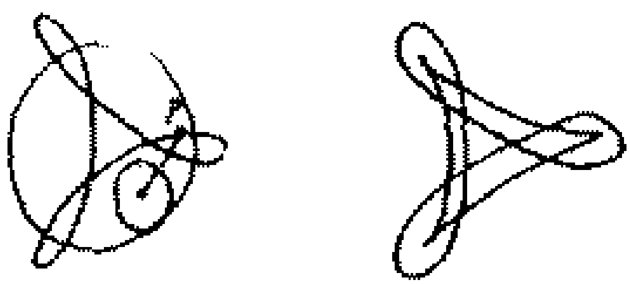
(5) Γ_r given by $z = re^{i\theta}$, $0 \le \theta < 2\pi$. $\therefore w = z^2 + 2c\bar{z} = r^2e^{2i\theta} + 2cre^{-i\theta}$.

$\therefore g_c\Gamma_r$ = epicycloid = locus of point P attached to (d r^2 from centre of) a disk of radius cr rolling inside a disk of radius $3cr$

Two cases

i) $r < c$
P inside disk

ii) $r > c$
P outside disk



Let $w = \rho e^{i\varphi}$

When $\theta = 0$ then $\varphi = 0$, $\rho = r^2 + 2cr \gtrless 3c^2$ as $r \gtrless c$.

$\therefore r < c \Rightarrow g_c\Gamma_r$ lies inside hypocycloid $g_c\Gamma = B_c$.

$r > c \Rightarrow g_c\Gamma_r$ begins outside & moves inside.

When $\theta = \frac{\pi}{3}$, and if $r < 2c$ then $\varphi = -\frac{\pi}{3}$ & $\rho = -r^2 + 2cr < c^2$

because $c^2 + r^2 - 2cr = (c-r)^2 > 0$
$\forall r \ne c$.

$\therefore$ as r moves away from c either increasing or decreasing, $g_c\Gamma_r$ moves away from $g_c\Gamma$ towards the origin in both cases, thus confirming $g_c\Gamma$ is a fold (except at cusp points).

⑤ (cont'd) If $r > c$ then $g_c\Gamma_r$ has self intersections ......

$f(\theta) = 0$ $\quad\therefore r^2\sin 2\theta + 2cr(-\sin\theta) = 0$

$\therefore 2\sin\theta\,(r\cos\theta - c) = 0$

$\therefore \sin\theta = 0$ or $\cos\theta = \frac{c}{r}$

$\Downarrow \qquad\qquad \Downarrow$

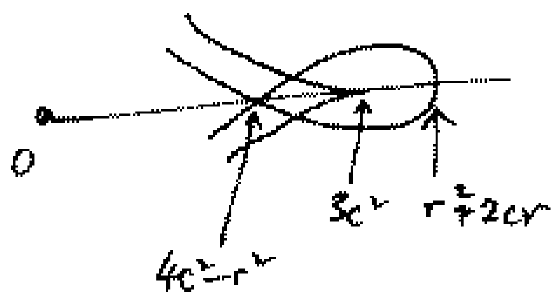
$\theta = 0$ $\qquad \theta = \pm\cos^{-1}\frac{c}{r}$, which $\exists$ because $r > c$

$\rho = r^2 + 2cr$ $\qquad \therefore \rho = r^2\cos 2\theta + 2cr\cos\theta$

$= r^2\left(2\frac{c^2}{r^2} - 1\right) + 2cr\frac{c}{r}$

$= 4c^2 - r^2$

$< 3c^2$



$\therefore g_c\Gamma_r$ has 3 self-intersections at $4c^2 - r^2$, $(4c^2 - r^2)e^{\pm\frac{2\pi i}{3}}$

---

⑥ $re^{i\theta} \xrightarrow{g_c} \rho e^{i\varphi}$

When $\theta = 0$ then $\varphi = 0$

$\therefore g_c$ maps line AB to A'B'.

By ④ g_c maps arc BC of Γ to arc B'C' of $g_c\Gamma$

When $\theta = \frac{2\pi}{3}$ then $\varphi = \frac{4\pi}{3}$, & g_c maps line CD to C'D'.

$\therefore g_c$ maps ABCD to A'B'C'D'.

Let Q = shaded region $\{z \mid r > c,\ 0 < \theta < \frac{2\pi}{3}\}$.

Then g_c maps Q homeomorphically (since Q contains no singularities) to one of the two regions bounded by A'B'C'D'.

Now $Q \ni 2ce^{\frac{\pi i}{3}}$, and g_c maps this point to 0.

$\therefore g_cQ$ is the region containing 0, shown shaded.

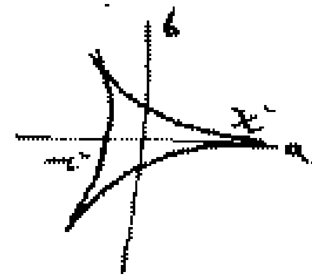
(3 … similarly g_c maps the other two regions outside Γ.

$\therefore$ g(outside Γ) covers $\begin{cases} \text{outside } g\Gamma \text{ once} \\ \text{inside } g\Gamma \text{ 3 times} \end{cases}$

Meanwhile g_c maps the inside of Γ homeomorphically (since it contains no singular points), to a region bounded by $g\Gamma$, with compact closure, which must be inside of $g\Gamma$.

$\therefore$ $g_c X$ covers $\begin{cases} \text{outside } g_c\Gamma \text{ once} \\ \text{inside } g_c\Gamma \text{ 4 times} \end{cases}$

(7) $\omega = f_c z = z^2 + 2\bar{z}$. When $z = c$, $\omega = 3c^2$; $z = -c$, $\omega = -c^2$.



$\therefore$ $f_c\Gamma$ meets a-axis in $\{3c^2, -c^2\}$

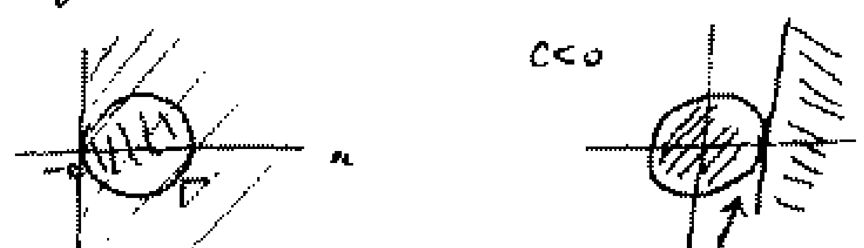
Bifurcation set $B = \bigcup g_c\Gamma \times c \subset (a,b,c)$ space.

$\therefore$ B meets (a,c)-plane in two parabolas $\begin{cases} a = 3c^2 \\ a = -c^2 \end{cases}$

As c grows, the hypocycloid grows parabolically $\propto c^2$.

(8) Hessian $H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2x+2c & -2y \\ -2y & -2x+2c \end{vmatrix} = 4(c^2 - x^2 - y^2)$.

H positive definite when $\begin{cases} x+c > 0 \\ H > 0 \end{cases}$ $\therefore$ $x^2+y^2 < c^2$



When $c<0$ the conditions have no intersection.
$\therefore$ M^* does not meet $(c<0)$.

When $c > 0$ they intersect interior of Γ.

$\therefore$ π_2 maps M^* diffeo onto $\left\{ \begin{matrix} x^2+y^2 < c^2 \\ c > 0 \end{matrix} \right\}$ cone.

$\therefore$ $\pi_1 M^* = \pi_1 (\pi_2 | M^*)^{-1}(\text{cone}) = f_c(\text{cone}) = (\text{interior } B) \cap (c>0)$

CATASTROPHE THEORY: Solution Sheet 5

(1) F_e given by $\begin{cases} x^5 - a - bx - cx^2 - dx^3 = 0 \\ 5x^4 - b - 2cx - 3dx^2 = 0 \end{cases}$

$$\therefore b = 4x^4 - 2cx - 3dx^2$$

$$\therefore a = x^5 - (5x^4 - 2cx - 3dx^2)x - cx^2 - dx^3$$
$$= -4x^5 + cx^2 + 2dx^3.$$

$$\dot{a} = -20x^4 + 2cx + 6dx^2$$
$$\dot{b} = 20x^3 - 2c - 6dx$$

$$\frac{da}{db} = \frac{\dot{a}}{\dot{b}} = -x.$$

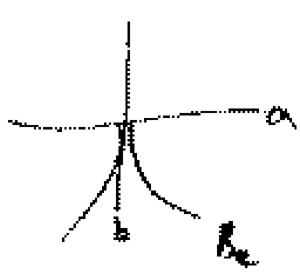
$(\dot{a}, \dot{b})$ gives the direction of the tangent to the curve at x.

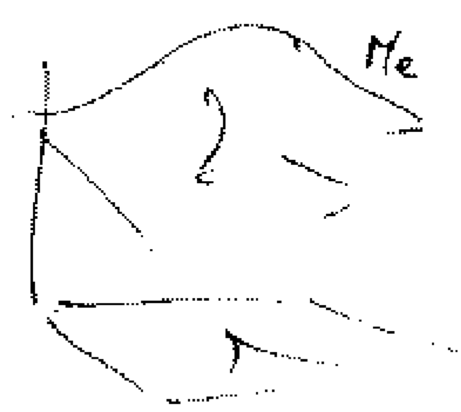
As the parameter x goes from $-\infty$ to ∞ the slope of the curve decreases monotonically from $+\infty$ to $-\infty$.

(2) Cusps given by $\frac{\partial^3 f}{\partial x^3} = 20x^3 - 2c - 6dx = 0.$

If $c = 0$ then $2x(10x^2 - 3d) = 0.$

If $d < 0$ then bracket > 0 $\therefore x = 0. \therefore a = b = 0$





like standard cusp.

(3) Fix $d<0$. Let $\theta_c = \frac{\partial^3 f}{\partial x^3} = 20x^3 - 2c - 6dx$.

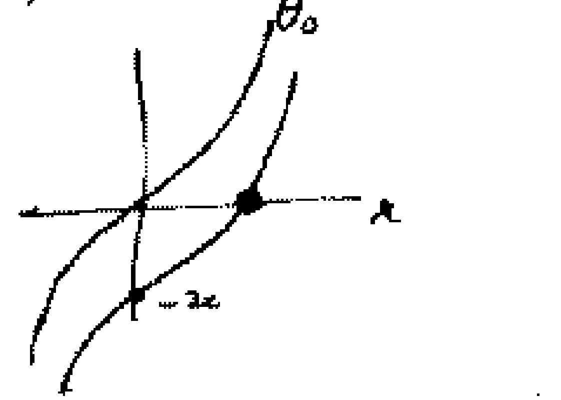
If $c=0$ then $\theta_0 = 20x^3 - 6dx$ = strictly monotonic because $d>0$

If $c>0$ then $\theta_c = \theta_0 - 2c$

$\therefore$ graph drops down by $2c$

$\therefore$ crosses x-axis once only.

$\therefore \exists$ only 1 cusp.



At that point $c = 10x^3 - 3dx$.

$$\therefore a = -4x^5 + (10x^3 - 3dx)x^2 + 2dx^3$$

$$= 6x^5 - dx^3$$

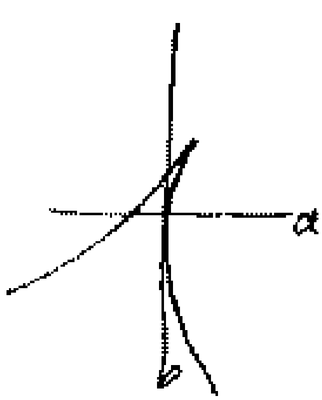
> 0 since $x>0$ & $d<0$.

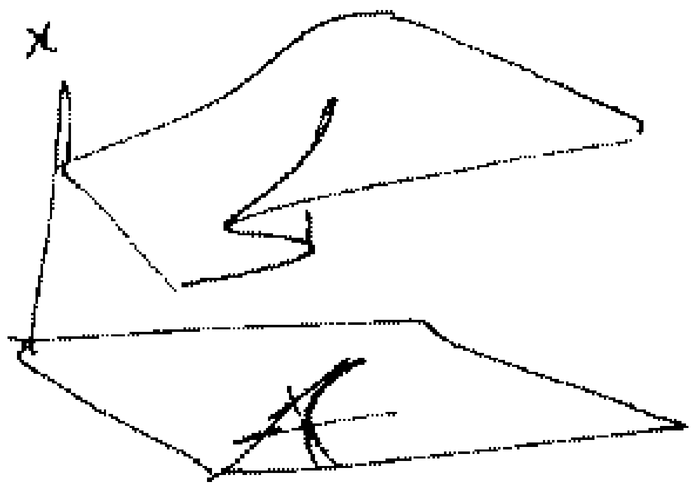
Also $b = 5x^4 - 2(10x^3 - 3dx)x - 3dx^2$

$$= -15x^4 + 3dx^2$$

< 0 since $x>0$ & $d<0$.

Small x : $(a,b) \sim (cx^2, -2cx)$ = parabola, axis +a-axis)





(4) $c=0$, $d=\frac{10\lambda^2}{3}$, $\lambda \neq 0$

$\frac{\partial^3 f}{\partial x^3} = 20x^3 - 20\lambda^2 x = 20x(x^2-\lambda^2) = 0$ when $x = 0, \pm\lambda$

$x=0$. $\underline{a = b = 0}$

$x=\lambda$. $a = -4\lambda^5 + 2\cdot\frac{10\lambda^2}{3}\cdot\lambda^3 = \underline{\frac{8}{3}\lambda^5}$

$b = 5\lambda^4 - 3\cdot\frac{10\lambda^2}{3}\cdot\lambda^2 = \underline{-5\lambda^4}$

$x=-\lambda$ $\underline{a = -\frac{8}{3}\lambda^5,\ b = -5\lambda^4}$

Double pt when $a=0$, $x \neq 0$. $\therefore -4x^5 + 2\frac{10\lambda^2}{3}x^3 = 0$.

$\therefore x^2 = \frac{5}{3}\lambda^2$. $x = \pm\sqrt{\frac{5}{3}}\lambda$

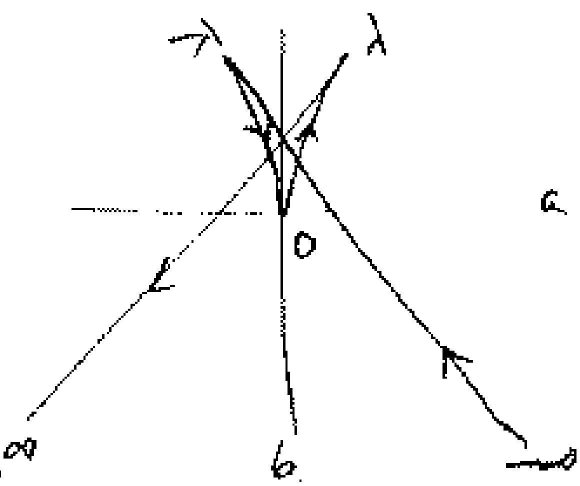
$\therefore b = 5\left(\sqrt{\frac{5}{3}}\lambda\right)^4 - 3\cdot\frac{10\lambda^2}{3}\cdot\frac{5}{3}\lambda^2$

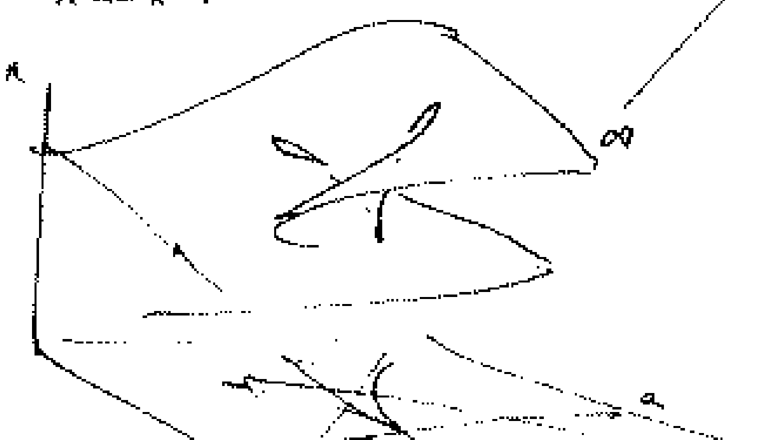
$= \frac{5\cdot 25}{9}\lambda^4 - \frac{50}{3}\lambda^4$

$= \frac{25}{9}(5-6)\lambda^4 = \underline{-\frac{25}{9}\lambda^4}$

$$\begin{cases} \dot a = -20x^4 + 20\lambda^2 x^2 = -20x^2(x^2-\lambda^2) \\ \dot b = 20x^3 - 20\lambda^2 x = 20x(x^2-\lambda^2) \end{cases}$$

	$\dot a$	$\dot b$
$x > \lambda$	$-$	$+$
$0<x<\lambda$	$+$	$-$
$-\lambda<x<0$	$+$	$+$
$x<-\lambda$	$-$	$-$





Undulation points given by $\frac{\partial^4 f}{\partial x^4} = 0$. $\therefore 60x^2 - 6d = 0$.

$$\therefore d = 10x^2. \qquad x^2 = \frac{d}{10}.$$

$$\frac{\partial^3 f}{\partial x^3} = 20x^3 - 2c - 6dx = 0.$$

$$\therefore c = 10x^3 - 3dx$$

$$= 10x^3 - 3 \cdot 10x^2 \cdot x$$

$$= -20x^3. \qquad \therefore x^3 = -\frac{c}{20}.$$

$$\therefore x^6 = \left(\frac{d}{10}\right)^3 = \left(-\frac{c}{20}\right)^2$$

$$\therefore 4\cancel{00}\, d^3 = 10\cancel{00}\, c^2$$

$$\therefore 5c^2 = 2d^3$$

If $c > 0$ then $x^3 < 0$. $\therefore x < 0$.

$$a = -4x^5 + cx^2 + 2dx^3 = -4x^5 + (-20x^3)x^2 + 2(10x^2)x^3$$

$$= (-4 \cancel{-20+20})x^5 = -4x^5$$

$$> 0$$

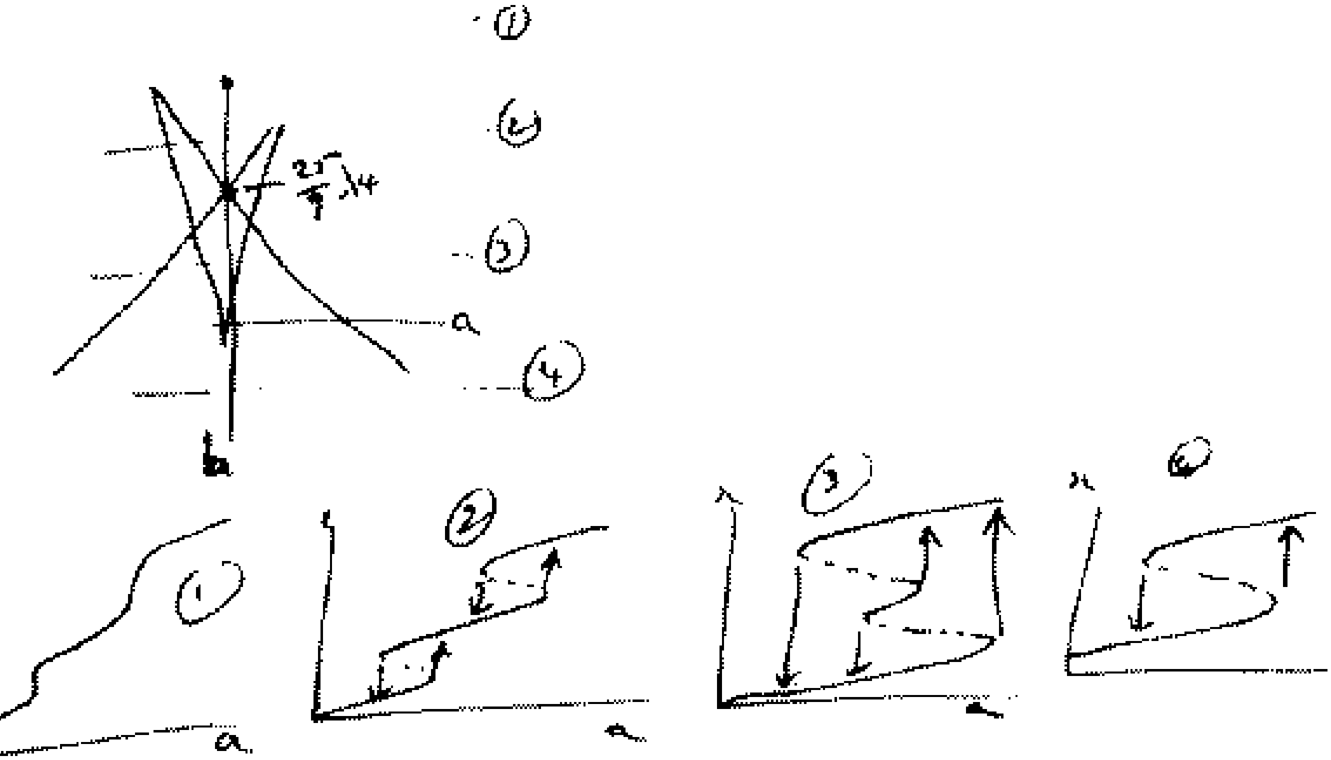
$$b = 4x^4 - 2cx - 3dx^2$$

$$= 4x^4 - 2(-20x^3)x - 3(10x^2)x^2$$

$$= (4 + 40 - 30)x^4$$

$$= 14x^4 > 0.$$

6



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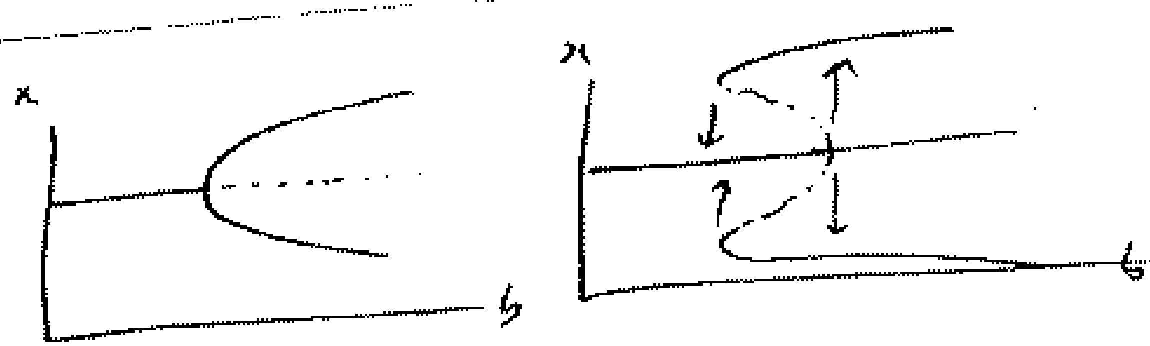

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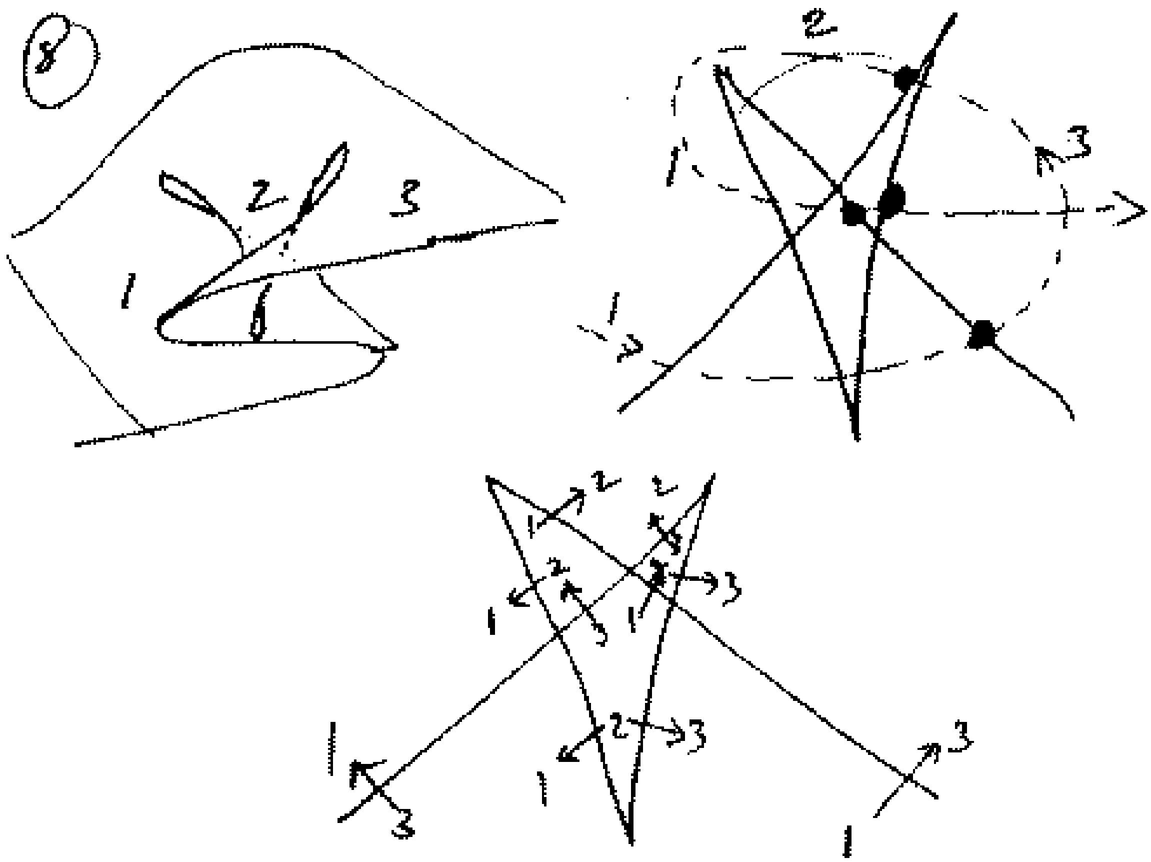
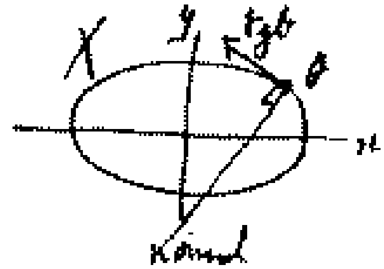


CATASTROPHE THEORY: Solution Sheet 6.

(1) $\left(\frac{x}{\alpha}\right)^2+\left(\frac{y}{\beta}\right)^2=1$, $\alpha>\beta>0$.



Let $\begin{cases} x = \alpha\cos\theta \\ y = \beta\sin\theta \end{cases}$, $0\le\theta<2\pi$.

Tangent $\begin{cases} \dot{x} = \frac{dx}{d\theta} = -\alpha\sin\theta \\ \dot{y} = \beta\cos\theta \end{cases}$

Normal $(x-\alpha\cos\theta)(-\alpha\sin\theta) + (y-\beta\sin\theta)\beta\cos\theta = 0$

$\therefore\ \alpha\sin\theta\, x - \beta\cos\theta\cdot y = (\alpha^2-\beta^2)\sin\theta\cos\theta$ — (1)

$\frac{d}{d\theta}$: $\alpha\cos\theta\, x + \beta\sin\theta\cdot y = (\alpha^2-\beta^2)(\cos^2\theta-\sin^2\theta)$ — (2)

(1) $\sin\theta$ + (2) $\cos\theta$: $\alpha x = (\alpha^2-\beta^2)\left[\sin^2\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\cos\theta\right] = (\alpha^2-\beta^2)\cos^3\theta$

$\therefore\ x = \left(\alpha - \frac{\beta^2}{\alpha}\right)\cos^3\theta$.

Symmetry $x\to y$, $\alpha\to\beta$, $\cos\theta\to\sin\theta$: $y = \left(\beta - \frac{\alpha^2}{\beta}\right)\sin^3\theta$.

$\therefore$ centre of curvature $(x,y) = \left(\left(\alpha-\frac{\beta^2}{\alpha}\right)\cos^3\theta, \left(\beta-\frac{\alpha^2}{\beta}\right)\sin^3\theta\right)$

$\therefore\ (\alpha x)^{2/3} = (\alpha^2-\beta^2)^{2/3}\cos^2\theta$

$(\beta y)^{2/3} = (\beta^2-\alpha^2)^{2/3}\sin^2\theta = (\alpha^2-\beta^2)^{2/3}\sin^2\theta$.

$\therefore\ (\alpha x)^{2/3} + (\beta y)^{2/3} = (\alpha^2-\beta^2)^{2/3}$. Evolute.

When $\theta=0$, centre of curvature of X at $(\alpha,0)$ is $\left(\alpha-\frac{\beta^2}{\alpha}, 0\right)$

$\therefore$ radius of curvature of X at $(\alpha,0)$ = $\frac{\beta^2}{\alpha}$

Along evolute $(\dot{x},\dot{y}) = \left(-\frac{\alpha^2-\beta^2}{\alpha}3\cos^2\theta\sin\theta, \frac{\beta^2-\alpha^2}{\beta}3\sin^2\theta\cos\theta\right)$

$= -\frac{3}{2}(\alpha^2-\beta^2)\sin 2\theta\left(\frac{\cos\theta}{\alpha}, \frac{\sin\theta}{\beta}\right)$

$= (0,0)$ iff $\sin 2\theta = 0$

$\therefore\ 2\theta = 0, \pi \pmod{2\pi}$

$\therefore\ \theta = 0, \frac{\pi}{2} \pmod{\pi}$

∴ cusps can only occur when $\theta = 0, \pi/2, \pi, 3\pi/2$.

To verify that these are cusps, let $x_0 = \alpha - \beta^2/\alpha$:

Hence for small θ, $x \doteq x_0\left(1-\theta^2/2\right)^3 \doteq x_0\left(1 - 3\theta^2/2\right)$

$$x - x_0 \doteq \frac{-3x_0}{2}\theta^2 \sim -\theta^2$$

$$y \doteq \left(\beta - \alpha^2/\beta\right)\theta^3 \sim -\theta^3.$$

$$\therefore (x-x_0, y) \sim (-\theta^2, \theta^3), \text{ cusp}$$

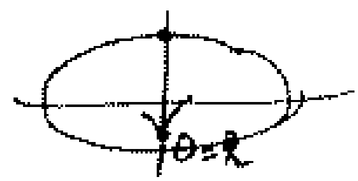
Similarly let $\theta = \pi/2 - \varphi$, for small φ:

$$\therefore x \doteq \left(\alpha - \beta^2/\alpha\right)\varphi^3 \sim \varphi^3$$

$$y \doteq y_0\left(1-\varphi^2/2\right)^3 \quad \text{where } y_0 = \beta - \alpha^2/\beta$$

$$\doteq y_0\left(1 - 3\varphi^2/2\right) \quad \therefore y - y_0 \doteq -\frac{3y_0\varphi^2}{2} \sim \varphi^2$$

$$\therefore (x, y-y_0) \doteq (\varphi^3, \varphi^2), \text{ cusp}$$



Cusps at $\theta = \pi, 3\pi/2$ by symmetry.

$$\text{Slope of evolute} = \frac{\dot y}{\dot x} = \frac{\frac{\beta^2-\alpha^2}{\beta}3\sin^2\theta\cos\theta}{-\frac{\alpha^2-\beta^2}{\alpha}3\cos^2\theta\sin\theta} = \frac{\alpha}{\beta}\tan\theta.$$

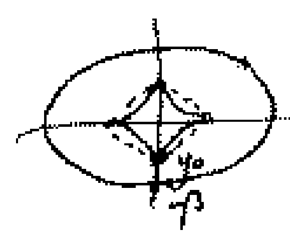
∴ as θ goes from 0 to $\pi/2$ slope increases from 0 to ∞.

∴ side concave.

By symmetry in x, y all 4 sides are concave.

∴ $E \subset$ quadrangle with vertices $(\pm x_0, 0), (0, \pm y_0)$

∴ $E \subset$ interior X $\iff$ $(0, y_0) \in$ interior X



$$\iff -\beta < y_0$$

$$\iff -\beta < \beta - \alpha^2/\beta$$

$$\iff \alpha^2/\beta < 2\beta$$

$$\iff \alpha^2 < 2\beta^2$$

$$\iff \alpha < \sqrt{2}\beta.$$

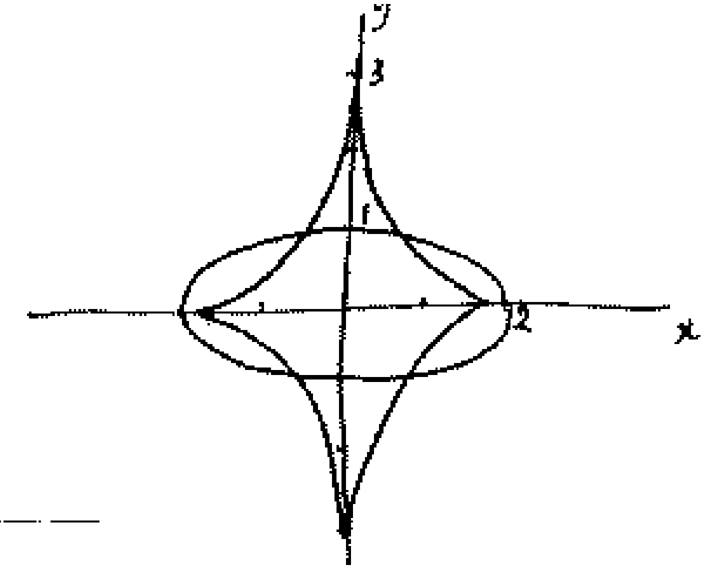
(i) $\alpha = 2$, $\beta = 1$.

$\therefore \alpha^2/\beta = 4$

$\beta^2/\alpha = 1/4$

$\therefore x_0 = \alpha - \beta^2/\alpha = 1\tfrac{3}{4}$

$y_0 = \beta - \alpha^2/\beta = -3$.



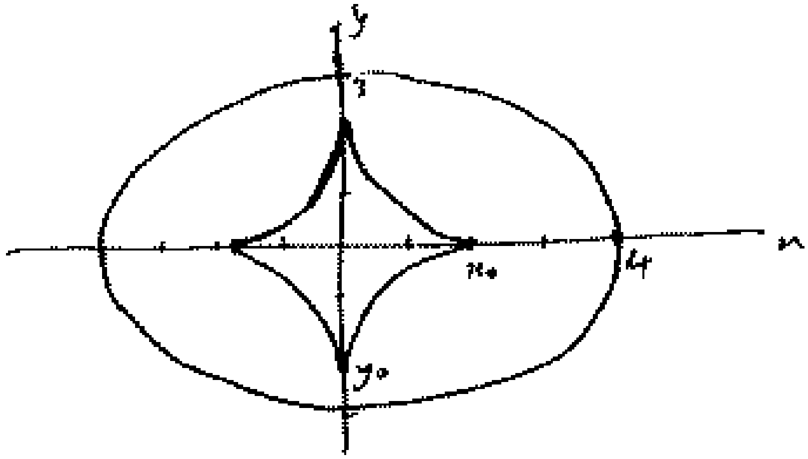
(ii) $\alpha = 4$, $\beta = 3$.

$\therefore \alpha^2/\beta = \frac{16}{3} = 5\tfrac{1}{3}$

$\beta^2/\alpha = 9/4 = 2\tfrac{1}{4}$

$\therefore x_0 = \alpha - \beta^2/\alpha = 1\tfrac{3}{4}$

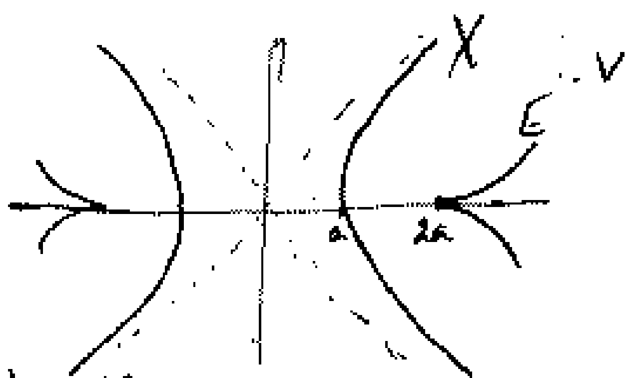
$y_0 = \beta - \alpha^2/\beta = -2\tfrac{1}{3}$



(2) $x^2 - y^2 = a^2$.

Let $\begin{cases} x = a\cosh\theta \\ y = a\sinh\theta \end{cases}$

$\therefore$ tangent $\begin{cases} \dot{x} = a\sinh\theta \\ \dot{y} = a\cosh\theta \end{cases}$



Normal $(x - a\cosh\theta)\cancel{a}\sinh\theta + (y - a\sinh\theta)\cancel{a}\cosh\theta = 0$ —(1)

$\therefore x\sinh\theta + y\cosh\theta = 2a\cosh\theta\sinh\theta$. —(2)

$x\cosh\theta + y\sinh\theta = 2a(\cosh^2\theta + \sinh^2\theta)$

(2) $\cosh\theta - (1)\sinh\theta$: $x = 2a\left[(\cosh^2\theta + \sinh^2\theta)\cosh\theta - \cosh\theta\sinh^2\theta\right]$

$= 2a\cosh^3\theta$.

Symmetry $\underset{\cosh \to \sinh}{x \to y}$: $y = 2a\sinh^3\theta$.

$\therefore x^{2/3} - y^{2/3} = (2a)^{2/3}(\cosh^2\theta - \sinh^2\theta) = (2a)^{2/3}$

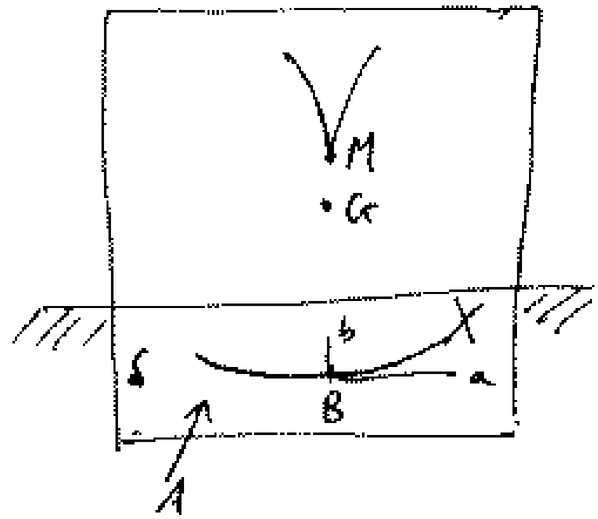
NOTE: I omitted the hypothesis that the square has side 1.

Floating implies $0 < \delta < 1$.

Beam $s = \frac{1}{2}$.

Area below water $A = \delta$.

$\therefore \rho = \frac{2s^3}{3A} = \frac{2/8}{3\delta} = \frac{1}{12\delta}$.



Buoyancy curve X of wall-sided ship:

$a^2 = 2\rho h$.

$\therefore a^2 = \frac{h}{6\delta}$.

Radius of curvature of X at $B = \rho = \frac{1}{12\delta}$.

$\therefore$ height of metacentre M above $B = \frac{1}{12\delta}$

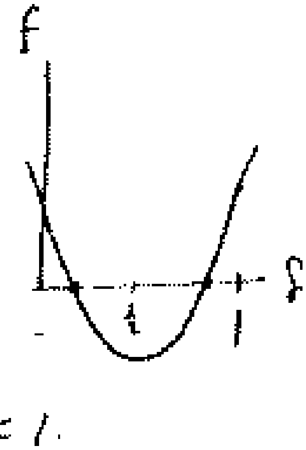
$\therefore$ " " above base $= \frac{\delta}{2} + \frac{1}{12\delta}$.

But " " c. of gravity G " " $= \frac{1}{2}$.

$\therefore$ metacentric height $\mu = \frac{\delta}{2} + \frac{1}{12\delta} - \frac{1}{2} = \frac{6\delta^2 - 6\delta + 1}{12\delta}$.

Let $f = 6\delta^2 - 6\delta + 1$.

$\therefore$ stability $\Leftrightarrow \mu > 0 \Leftrightarrow f > 0$.



Now $f = 0$ when $\delta = \frac{3 \pm \sqrt{9-6}}{6} = \frac{3 \pm \sqrt{3}}{6}$

$\therefore f > 0$ when $\delta < \frac{3-\sqrt{3}}{6}$ or $> \frac{3+\sqrt{3}}{6}$.

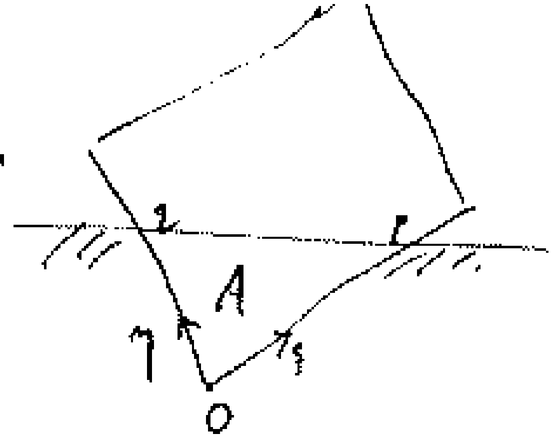
Combine this with $0 < \delta < 1$.

$\therefore$ stability $\Leftrightarrow 0 < \delta < \frac{3-\sqrt{3}}{6}$ or $\frac{3+\sqrt{3}}{6} < \delta < 1$.

To find out what happens when $\frac{3-\sqrt{3}}{6} < \delta < \frac{3+\sqrt{3}}{6}$

we first check stability at $45°$.

Suppose $0<\delta<\tfrac12$.



Take axes ξ,η as shown

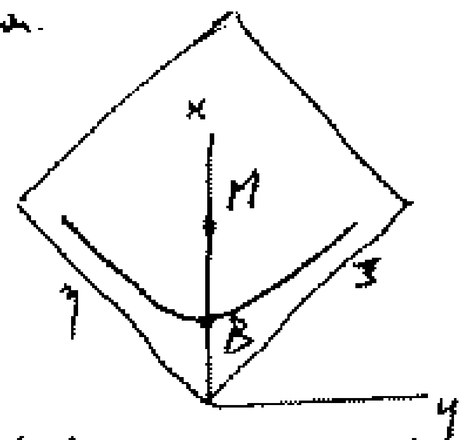
Suppose waterline goes from $(p,0)$ to $(0,q)$

$\therefore A = \tfrac12 pq = \delta$.

Centre of buoyancy: $(\bar\xi,\bar\eta) = \left(\frac{p}{3}, \frac{q}{3}\right)$ (centre of gravity of triangle)

$\therefore$ buoyancy locus X: $\xi\eta = \frac{pq}{9} = \frac{2\delta}{9}$ rectangular hyperbola

Now take axes x,y at 45° as shown.



$\therefore \xi = \frac{x+y}{\sqrt2},\ \eta = \frac{x-y}{\sqrt2}$.

$\therefore \xi\eta = \frac{x^2-y^2}{2} = \frac{2\delta}{9}$

$\therefore x^2-y^2 = \frac{4\delta}{9} = \left(\frac{2\sqrt\delta}{3}\right)^2$

$\therefore$ at 45° c. of buoyancy B is $(x,y) = \left(\frac{2\sqrt\delta}{3}, 0\right)$

metacentre M = c. of curvature $= \left(\frac{4\sqrt\delta}{3}, 0\right)$ by question 2.

$\therefore$ Centre of gravity $G = \left(\frac{1}{\sqrt2}, 0\right)$.

Metacentric height $\mu = \frac{4\sqrt\delta}{3} - \frac{1}{\sqrt2}$.

$\therefore$ Stability at $45^\circ \iff \mu>0$

$\iff \frac{4\sqrt\delta}{3} > \frac{1}{\sqrt2}$

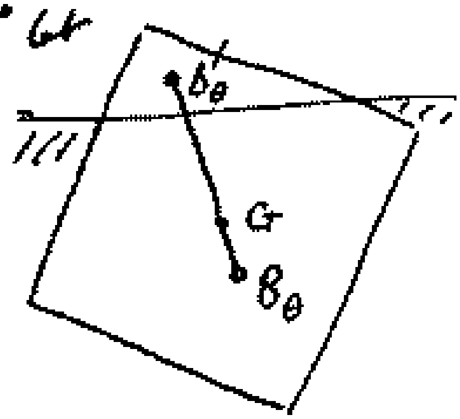
$\iff \sqrt\delta > \frac{3}{4\sqrt2}$

$\iff \delta > \frac{9}{32}$

Now suppose $\frac{1}{2} < \delta < 1$. Near $\theta = 45°$ let

X = locus of c. of buoyancy B_θ

X' = " " c. of gravity B'_θ of part above water.



Then G lies on $B_\theta B'_\theta$ and

$$\frac{GB_\theta}{GB'_\theta} = \frac{1-\delta}{\delta}$$

By above locally X' = rectangular hyperbola.

$\therefore$ X = similar rect. hyperbola, reflected in G, & shrunk by $\frac{1-\delta}{\delta}$.

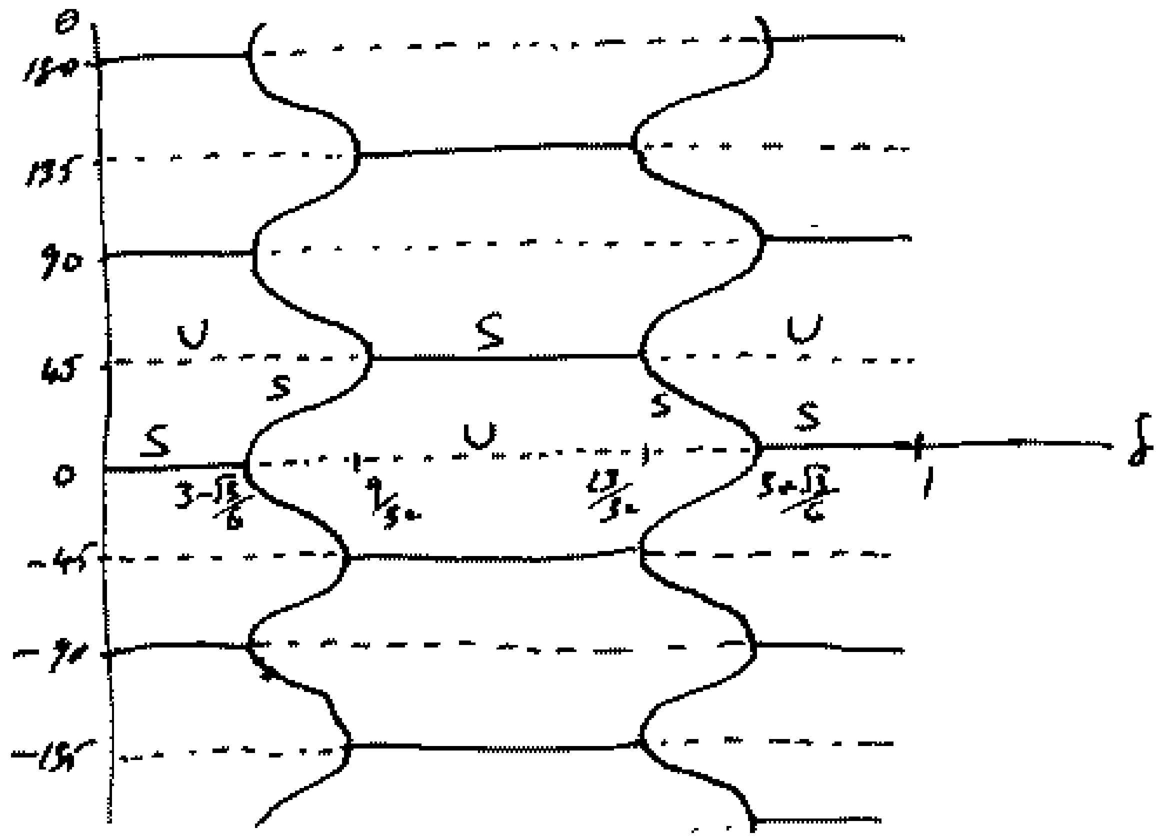
$\therefore$ $45°$ is stable $\iff$ G lies outside evolute E of X

$\iff$ " " " E' of X'

$\iff$ $1-\delta > 9/32$, by above

$\iff$ $\delta < 1 - 9/32 = 23/32$

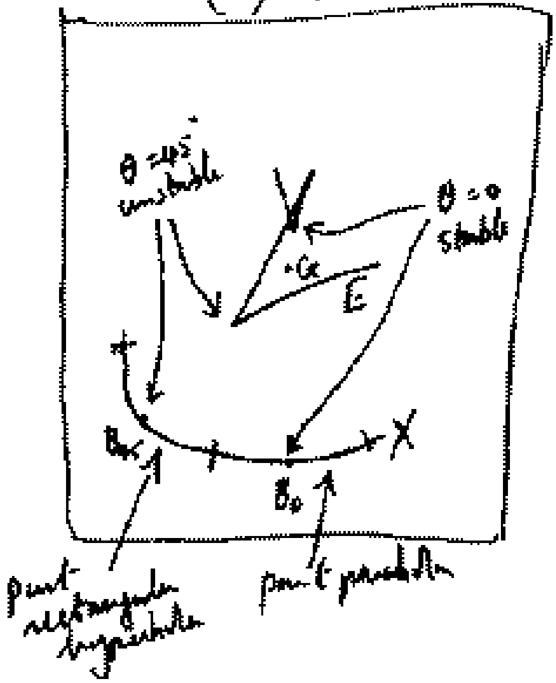
As δ increases from $\frac{3-\sqrt{3}}{6}$ to $9/32$ the stable angle θ increases from 0 to $45°$ (& symmetrically decreases from 0 to $-45°$)

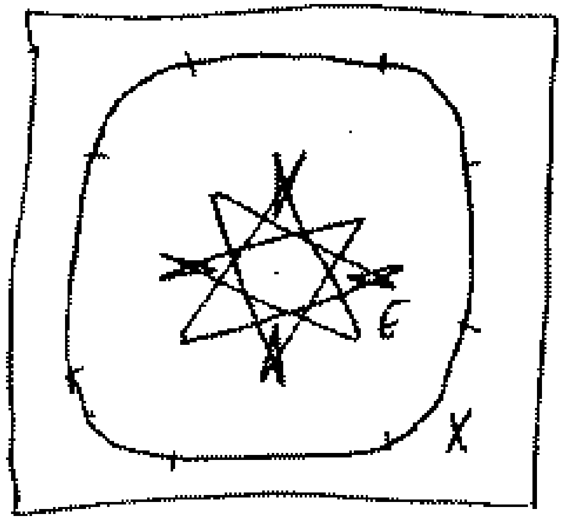


Part of bifurcation curves & umbilic torus E near (0,45)

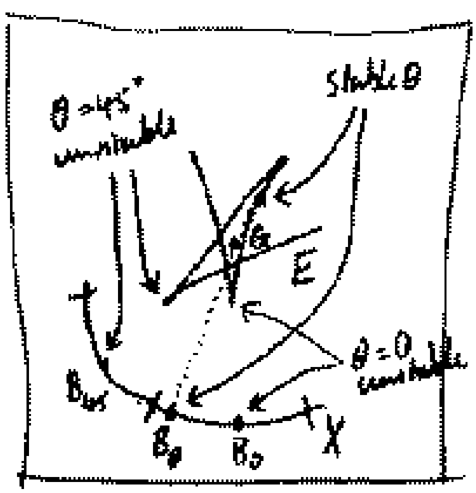
Complete bifurcation curves & umbilic torus

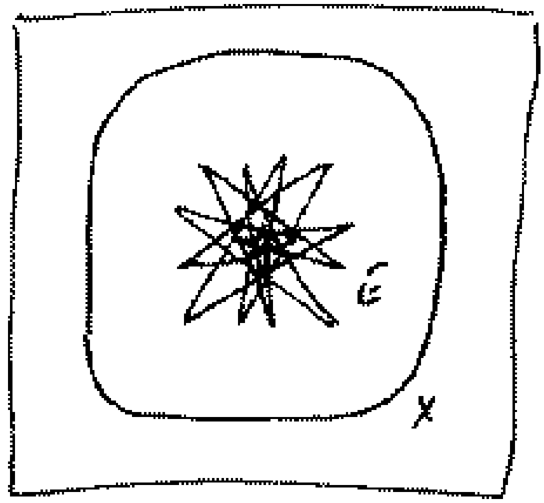
$0 < \delta < \frac{3-\sqrt{3}}{6}$



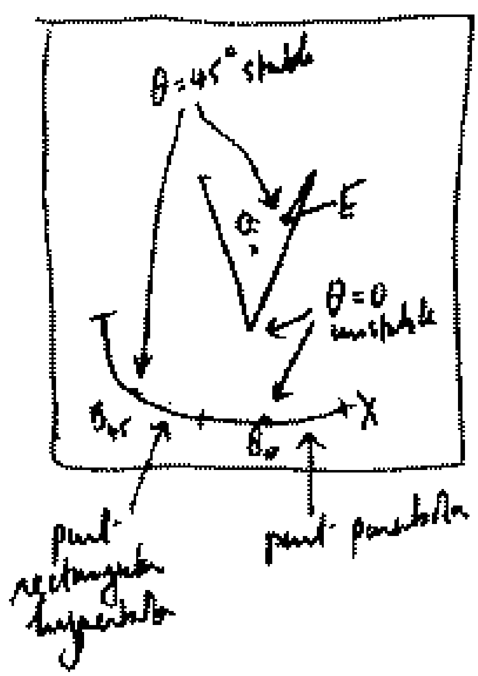


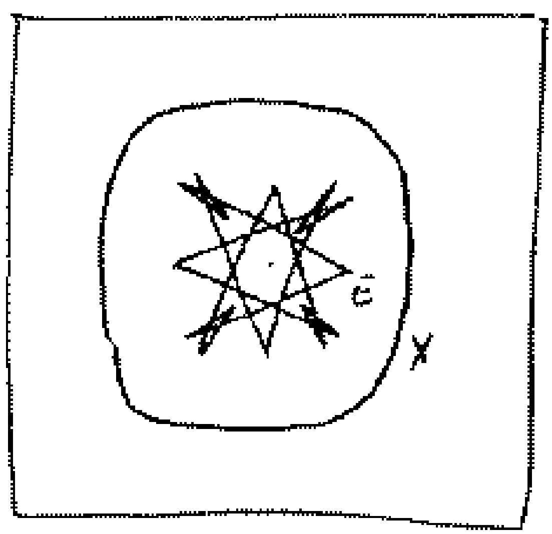
$\frac{3-\sqrt{3}}{6} < \delta < \frac{9}{32}$





$\frac{9}{32} < \delta < \frac{23}{32}$



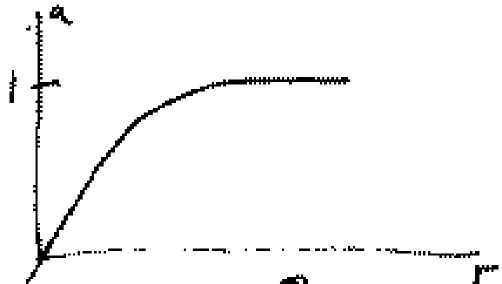


CATASTROPHE THEORY: Solution sheet 7.

(1)

For $r>0$, $\frac{da}{dr} = 1 - e^{-\frac{1}{r}}\left(1+\frac{1}{r}\right) \xrightarrow[r\to 0]{} 1$

For $n \geq 2$, $\frac{d^n a}{dr^n} = e^{-\frac{1}{r}}\left(\text{polynomial in } \frac{1}{r}\right) \xrightarrow[r\to 0]{} 0$



$\therefore$ all partial derivative $\exists$ & are continuous. $\therefore C^\infty$.

For $r\leq 0$, $\frac{da}{dr} = 1$. For $r>0$ $\quad 1+\frac{1}{r} < e^{\frac{1}{r}}$

$\therefore \left(1+\frac{1}{r}\right)e^{-\frac{1}{r}} < 1$.

$\therefore \frac{da}{dr} > 0$.

$\therefore$ monotonic increasing.

For $r>0$, $\quad 1-\frac{1}{r} < e^{-1/r} < 1-\frac{1}{r}+\frac{1}{2r^2}$

$\therefore \frac{1}{r} > 1-e^{-1/r} > \frac{1}{r} - \frac{1}{2r^2}$

$\therefore 1 > a(r) > 1 - \frac{1}{2r}$

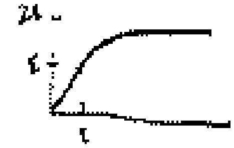
$\therefore a(r) \to 1$ as $r \to \infty$.

For $r \leq \varepsilon$ $\quad a\left(\frac{r}{\varepsilon}-1\right) = \frac{r}{\varepsilon}-1$. $\quad \therefore b(r) = \varepsilon\left(\frac{r}{\varepsilon}\right) = r$.

$\therefore b$ keeps $[0,\varepsilon]$ pointwise fixed.

$b\uparrow$ because a is.

$b(r) \to \varepsilon(1+1) = 2\varepsilon$ as $r\to\infty$.



$\therefore b[0,\infty) = [0,2\varepsilon)$.

$\therefore b$ diffeo.

Let $x=(r,\theta)$ in polar coords, $r \geq 0$, $\theta\in S^{n-1}$.

$\therefore c(r,\theta) = (b(r),\theta)$

$\therefore c\mathbb{R}^n = B_{2\varepsilon}$ diffeo & c keeps B_ε pointwise fixed.

(2) We prove $\dim(E/m^{k+1}) = \frac{n+k!}{n!\,k!}$ by induction.

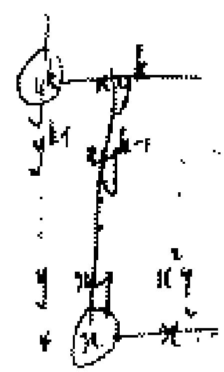
Firstly, if $n=0$ then $E=\mathbb{R}$, $m=0$, & so both sides $=1$, $\forall k$.
Next, if $k=0$ then $E/m=\mathbb{R}$, ... , $\forall n$.
$\therefore$ result true for $n+k<2$, [Here $n,k\geq 0$].
Assume true for $n+k<N$, where $N\geq 2$.
Given $n+k=N$, $n>0$, $k>0$, then

E/m^{k+1} = polynomials of degree $\leq k$ in $x_1,\dots,x_n$
= (" " " " " $x_1,\dots,x_{n-1}$)
+ (" " " $\leq k-1$ " $x_1,\dots,x_n$)

$$= \frac{n+k-1!}{n-1!\,k!} + \frac{n+k-1!}{n!\,k-1!} \text{ by induction \& cases } n=0 \text{ \& } k=0$$

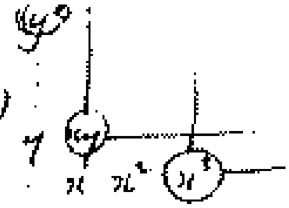
$$= \frac{n+k!}{n!\,k!}$$

(3) (i) $x^2+y^k,\ k\geq 2$ $J=(2x, ky^{k-1})=(x,y^{k-1})\supset m^{k-1}$
$\therefore mJ\supset m^k$ $\therefore$ k-det



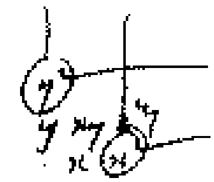
Not $(k-1)$-det because $(k-1)$-jet $=x^2$, isolat.
$\therefore$ determinacy $=k$.
Base of $m/J = y, y^2, \dots, y^{k-2}$ $\therefore$ codim $=k-2$
Unfolding $= x^2+y^k+a_1y+a_2y^2\dots+a_{k-2}y^{k-2}$.

(ii) $x^2y+y^k,\ k\geq 3$ $J=(2xy, x^2+ky^{k-1})$
$\therefore mJ = (x^2y, xy^2, x^3+kxy^{k-1}, x^2y+ky^k)$
$= (x^2y, xy^2, x^3, y^k)$
$\supset m^k$ $\therefore$ k-det
Not $(k-1)$-det because $(k-1)$-jet $=x^2y$, isolat.
$\therefore$ determinacy $=k$.

Base for $m/J = x, y, \dots, y^{k-1}$ (this indicates x^2) 

$\therefore$ codim $= k$ & unfolding $= x^2y + y^k + a_0x + a_1y + \dots + a_{k-1}y^{k-1}$

(iii) x^3+y^3. $\quad J = (3x^2, 3y^2)$

$\therefore mJ = (x^3, x^2y, xy^2, y^3) = m^3$ 

$\therefore$ 3-det. Not 2-det because 2-jet $= 0$.

$\therefore$ determinacy $= 3$

Base $m/J = x, y, xy$. $\quad \therefore$ codim $= 3$

unfolding $= x^3+y^3+ax+by+cxy$

(iv) x^3+y^4 $\quad J = (3x^2, 4y^3)$

$mJ = (x^3, x^2y, xy^3, y^4) \supset m^4$

$\therefore$ 4-det. Not 3-det because 3-jet $= x^3$, indet.

$\therefore$ determinacy $= 4$

Base $m/J = x, y, xy, y^2, xy^2$. $\quad \therefore$ codim $= 5$

& unfolding $= x^3+y^4+ax+by+cxy+dy^2+exy^2$

(4) Given $f: \mathbb{R}^n \to \mathbb{R}$, $f \in m^2$, $r = \operatorname{rank} j^2 f$, f k-det, $k \geq 3$ $\Longrightarrow$ $\exists$ coordinates of $\mathbb{R}^n$ such that $f = g + h$, where

$g = \pm x_1^2 \pm x_2^2 \pm \dots \pm x_r^2$

$h =$ polynomial in $x_{r+1}, \dots, x_n$ of degree ≥ 3, $\leq k$

Proof. Expand f in Taylor series $f = \underbrace{f_0 + f_1}_{0 \text{ because } f \in m^2} + f_2 + \cdots$

Diagonalise f_2 by a linear change of coordinates & make $f_2 = g$. Suppose, inductively, that x_1 does not appear in $f_3, \dots, f_{q-1}$, $q \geq 3$. We can kick x_1 out of f_q by a non-linear change of coords. as follows: Put $f_q = A \pm 2x_1 B$, where $A =$ all terms $\not\ni x_1$, $\pm$ as sign of x_1^2 in g.

Put $y_1 = x_1 + B$.

$\therefore y_1^2 = x_1^2 \pm 2x_1B + B^2$

∴ degree $k^2 = (q-1)^2 \geq 2(q-1)$, since $q \geq 3$

$= q + (q-2)$

$\geq q+1$, since $q \geq 3$

Substitute y_1 for x_1 & we have kicked y_1 out of f_2.

Similarly kick $x_1, \dots, x_r$ out of $f_2, \dots, f_k$.

By k-determinacy $f \sim$ its k-jet = germ as above.

---

(5) $f = x^2 + 2xy^2 = (x+y^2)^2 - y^4$

$= \xi^2 - y^4$ putting $\xi = x+y^2$.

∴ dual cusp.

$J = (\xi, y^3) \supset m^3$ ∴ $mJ \supset m^4$ ∴ 4 det.

Not 3-det because 3-jet $= \xi^2$, not det. ∴ determinacy = 4

m/J base y, y^2. ∴ codim = 2. Unfolding $= \xi^2 - y^4 + ay + by^2$

$= x^2 + 2xy^2 + ay + by^2$

---

(6) $g = x^2 + 2xy^2 + y^4 + y^5$ has same 3-jet as f.

$= (x+y^2)^2 + y^5$

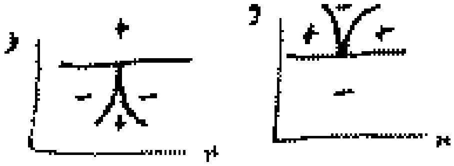
$= \xi^2 + y^5$, putting $\xi = x+y^2$

= swallowtail of cusp.

---

(7) $x^2y + y^4 \sim -x^2y + y^4$ by diffeo $(x,y) \mapsto (x,-y)$

$x^2y + y^4 \not\sim x^2y - y^4$ because LHS is positive inside a cusp, while RHS is negative inside a cusp,



& ∄ diffeo of R^2 sending + region of one into + region of the other.

---

(8) Theorem $m^{k+1} \subset m^2 J \Rightarrow f$ k-determinate.

This is a slight sharpening of the theorem in the lecture notes, & requires only a sharpening of one lemma from

$m^k \subset mJ \Rightarrow m^k \subset m\Omega$

to $m^{k+1} \subset m^2 J \Rightarrow m^{k+1} \subset m^2 \Omega$, as follows.

Here J = Jacobian ideal of f in E

f' has same k-jet as f

$F = (1-t)f + tf'$.

$\mathcal{A}$ = ring of germs at $(0,t_0)$ of functions $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

$\mathfrak{a}$ = maximal ideal of $\mathcal{A}$

Ω = Jacobian ideal of F in $\mathcal{A}$.

Proof $f = F + t(f - f')$

Now $f - f' \in m^{k+1}$ $\therefore \frac{\partial}{\partial x_i}(f-f') \in m^k$.

$\therefore \frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i} + t\frac{\partial}{\partial x_i}(f-f') \in \Omega + \mathcal{A}m^k$.

$\therefore J \subset \Omega + \mathcal{A}m^k$.

$\therefore m^{k+1} \subset m^2 J \subset m^2\Omega + \mathcal{A}m^{k+2}$

hypothesis

$\subset m^2\Omega + \mathfrak{a}(\mathcal{A}m^{k+1})$, since $m \subset \mathfrak{a}$.

$\therefore \mathcal{A}m^{k+1} \subset m^2\Omega + \mathfrak{a}(\mathcal{A}m^{k+1})$, since $\mathcal{A}\Omega = \Omega$, $\mathcal{A}^2 = \mathcal{A}$.

$\therefore \mathcal{A}m^{k+1} \subset m^2\Omega$ by Nakayama's lemma, since $\mathcal{A}m^{k+1}$ is a finitely generated $\mathcal{A}$-module, generated by monomials in x_i of degree $k+1$.

$\therefore m^{k+1} \subset \mathcal{A}m^{k+1} \subset m^2\Omega$.

---

(9) (i) $x^4 + y^4$ $\qquad J = (4x^3, 4y^3) = (x^3, y^3)$

$\therefore mJ = (x^4, x^3y, xy^3, y^4)$

$\therefore m^2J = (x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5) = m^5$.

$\therefore$ 4-det. Not 3-det because 3-jet $= 0$.

$\therefore$ determinacy $= 4$

Remark $mJ \not\ni x^2y^2$.

$\therefore mJ \not\supset m^4$

$\therefore$ weaker result would have been insufficient.

Base m/J = $\boxed{\begin{matrix} y^2 & xy^2 & x^2y^2 \\ y & xy & x^2y \\ & x & x^2 \end{matrix}}$ $\qquad \therefore$ codim $= 8$

Unfolding $= x^4 + y^4 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^2y + a_7xy^2 + a_8x^2y^2$

$J = (x^4, y^4)$

$\therefore mJ = (x^5, x^4y, xy^4, y^5)$

$\therefore m^2J = (x^6, x^5y, x^4y^2, x^2y^4, xy^5, y^6) \supset m^7$

$\therefore$ 6 det.

Now $mJ \not\ni x^3y^3$ $\therefore$ $mJ \not\supset m^6$ $\therefore$ not 5-det

SURPRISE!

$\therefore$ determinacy $= 6$.

Base m/J =

y^3	xy^3	x^2y^3	x^3y^3
y^2	xy^2	x^2y^2	x^3y^2
y	xy	x^2y	x^3y
	x	x^2	x^3

$\therefore$ codim $= 16 - 1 = 15$

Unfolding $= x^5 + y^5 + a_1 x + \dots + a_{15} x^3y^3$

$x^3+y^3+z^3$

$J = (x^2, y^2, z^2)$

$\therefore m^2J$ = (all monomials of form x^4, x^3y, x^2y^2) $= m^4$.

$\therefore$ 3-det. Not 2-det because 2-jet $= 0$.

$\therefore$ determinacy $= 3$

Base m/J $= x, y, z, xy, yz, zx, xyz$.

$\therefore$ codim $= 7$. Unfolding $= x^3+y^3+z^3 + a_1x + \dots + a_7 xyz$.

---

(10). Paradox: $f_k = j^kf - j^{k-1}f$ = invariant.

However this is not a permissible equation because the terms lie in different groups & therefore cannot be subtracted.

$f_k \in E$, $j^kf \in E/m^{k+1}$, $j^{k-1}f \in E/m^k$.

Example let $f = x = y + y^2$, under the diffeo change of coords $x = y + y^2$.

Then $j^1f = x = y \in E/m^2$, because $y^2 = 0$ in this group.

$j^2f = x = y + y^2 \in E/m^3$.

But the second term of Taylor Series $f_2 = \begin{cases} 0 & \text{in } x\text{-coordinates} \\ y^2 & \text{in } y\text{-coordinates} \end{cases}$

Since these are not equal, f_2 is not invariant.

REPRINTS BY E.C. ZEEMAN

1. Catastrophe theory: Draft for a Scientific American article, *Catastrophe Theory: Selected papers 1972-1977*, Addison-Wesley, Reading, Mass. 1977, 1-64; & (shortened version) Sc. Am. 234 (1976) 4, 65-83.

2. Levels of structure in catastrophe theory, *op. cit.*, 65-79; & Proc. Int. Cong. Math. Vancouver (1974) 2, 533-546.

3. Differential equations for the heartbeat and nerve impulse, *op. cit.*, 81-140; & *Towards a theoretical biology*, (ed. C.H. Waddington) Edinburgh Univ. Press 4 (1972), 8-67.

4. Primary and secondary waves in developmental biology, *op. cit.*, 141-234; & Lectures on Maths in the Life Sciences 7 (1974), Amer. Math. Soc., Rhode Island, 69-161.

5. (with J. Cooke) A clock and wavefront model for the control of repeated structures during animal morphogensis, *op. cit.*, 235-256; & J. Theoretical Biology 58 (1976), 455-476.

6. (Addendum by R. Bellairs) Gastrulation and formation of somites in amphibia and birds, *op. cit.*, 257-266; & Structural Stability, the theory of catastrophes and applications in the sciences, Springer Lecture Notes in Maths. 525 (1976) 396-401.

7. Dialogue between and Biologist and a Mathematician, *op. cit.*, 267-286; & Biosciences Communications (1977).

74, 76

8. Brain modelling, *op. cit.*, 287-292; & Structural Stability, the theory of catastrophes and applications in the sciences, Springer Lecture Notes in Maths. 525 (1976) 367-372.

9. Duffing's equation in brain modelling, *op. cit.*, 293-301; & Bull. Inst. Math. & Appl. 12 (1976), 207-214.

10. (with C.A. Isnard) Some models in the social sciences, *op. cit.*, 303-360; & *Use of models in the social sciences* (ed. L. Collins), Tavistock, London (1975), 44-100.

11. On the unstable behaviour of stock exchanges, *op. cit.*, 361-372; & J. Math. Economics 1 (1974), 38-49.

12. Conflicting judgements caused by stress, *op. cit.*, 361-372; & Br. J. maths & stats. Psychology 29 (1976), 19-31.

13. (with C.S. Hall, P.J. Harrison, G.H. Marriage, P.H. Shapland) A model for institutional disturbances, *op. cit.*, 387-402; & Br. J. maths & stats. Psychology 29 (1976), 66-80.

14. Prison disturbances, *op. cit.*, 403-407; & Structural Stability, the theory of catastrophes and applications in the sciences, Springer Lecture Notes in Maths. 525 (1976) 402-406.

15. A catastrophe machine, *op. cit.*, 409-416; & *Towards a theoretical biology*, (ed. C.H. Waddington) Edinburgh Univ. Press 4 (1972), 276-282.

16. Euler buckling, *op. cit.*, 417-440; & Structural Stability, the theory of catastrophes and applications in the sciences, Springer Lecture Notes in Maths. 525 (1976) 373-395.

17. Stability of ships, *op. cit.*, 441-495; Proc. Esc. Lat. Am. Math. 3 (1976).

18. (with D.J.A. Trotman) The classification of elementary catastrophes of codimension ≤5, *op. cit.*, 497-562; & Structural Stability, the theory of catastrophes and applications in the sciences, Springer Lecture Notes in Maths. 525 (1976) 263-327.

19. The umbilic bracelet and the double-cusp catastrophe, *op. cit.*, 563-603; & Structural Stability, the theory of catastrophes and applications in the sciences, Springer Lecture Notes in Maths. 525 (1976) 328-366.

20. Research ancient and modern, *op. cit.*, 605-614; & Bull. Inst. Math. & Appl. 10 (1974) 272-281.

21. (with R. Thom) Catastrophe theory: its present state and future perspectives, *op. cit.*, 615-650; & *Dynamical Systems, Warwick 1974* (ed. A.K. Manning), Springer Lecture Notes in Maths 468, 366-401.

22. Afterthought, *op. cit.*, 651-658.

23. A catastrophe theory of anorexia nervosa, *Hunger Models* (ed. D.A. Booth), Academic Press, London (1978) 407-422.

24. Applications of catastrophe theory, *Manifolds-Tokyo 1973*, Univ. of Tokyo Press (1975) pp.11-23.

25. A geometrical model of ideologies, *Transformations: mathematical approaches to culture change* (eds. C. Renfrew & K.L. Cooke), Academic Press (1979) 463-479.

26. Catastrophe theory, *Proceedings of the Royal Institution of Great Britain*, vol 49, (1976) pp. 77-92.

27. Catastrophe theory, Laudatio for Professor R. Thom, *Structural stability in physics* (ed. W. Guttinger & H. Eikemeir), Springer Series in Synergetics (1979) 12-22.

28. (with C.S. Hall, P.J. Harrison, G.H. Marriage & P.H. Shapland) A model for prison disturbances, *Brit. J. Criminology*, 17 (1977) 251-263.

29. (with J.Q. Smith & P.J. Harrison) The analysis of some discontinuous decision processes, *Euro. J. Op. Res.* 7 (1981) 30-43.

30. Population dynamics from game theory, *Global Theory of Dynamical Systems* (ed. Z. Nitecki & R.C. Robinson), Springer Lecture Notes in Maths. 819 (1980) 471-497.

31. Dynamics of the evolution of animal conflicts, *J. Theor. Biology* 89 (1981) 249–270.

32. (with M. Blad) Oscillations between repressed inflation and Keynsian equilibrium due to inertia in decision making, *J. Econ. Theory* 28 (1982) 165-182.

33. Bifurcation, catastrophe and turbulence, *New directions in applied mathematics*, (ed. P.J. Hilton & G.S. Young) Springer Verlag (1981) 109-153.

34. Catastrophe models in administration, *Assoc. Inst. Res. Proc.* 3 (1980) 9-24.

35. (with J. Hayden) A Lagrangian Klein bottle, *Math. Proc. Camb. Phil. Soc.* 89 (1980) 193-200.

36. Decision making and evolution, *Theory and explanation in archaeology* (eds. C. Renfrew, M.J. Rowlands & B.A. Segraves) Academic Press, New York (1981) 315-346.

37. (with C. Glenton) Bifurcation and catastrophe theory, *Contemporary Mathematics*, Volume 9 (1982) Am. Math. Soc., Providence, 207-272.

38. (with B. Wetherilt) Bibliography on Catastrophe Theory, Warwick reprint 1981.

39. Sudden changes of perception, *Logos et Théorie des Catastrophes* (ed. J. Petitot) 1983, Cerisy (to appear).

40. On the psychology of a hijacker, *Analysing Conflict and its Resolution* (ed. P.G. Bennett) IMA New Series 8, OUP (1987), 71-91.

41. (with G. Bonanno) Limited knowledge of demand and oligopoly equilibria, *Jour. Econ. Theory*, 35 (1985) 276-283.

42. Dynamics of Darwinian evolution, *Dynamical systems: a renewal of mechanism* (ed. S. Diner, D. Fargue & G. Lochak), World Sc. Pub. Co., Singapore (1986) 155-165.

43. Gears from the Greeks, *Proc. Roy. Instn Gt. Brit.* 58 (1986) 139-156.

44. *Geometry and Perspective, Royal Institution Mathematics Masterclass* (video and book), B.B.C. Open Univ. Prod. Unit, and Dept. Educ. & Sci., 1987.

45. Stability of dynamical systems, *Nonlinearity* 1 (1988) 115-155.

46. (with S. López de Medrano & C.H. Watts) Stable diffeomorphisms are dense, Warwick University Preprint (1988).

47. (with M. Chaperon, S. López de Medrano & C.H. Watts) Almost invariant probability measures for diffeomorphisms and flows on a compa[illegible] Riemannian manifold, and the associated notion of structural stability, *Comptes Rendus* 1988 (to appea[illegible]).

48 On the classification of dynamical systems, Bull. Lond. Math Soc. 20 (1988), 545–

49 Gyros & boomerangs: RI Math Masterclass, BBC Open Univ & DES, 1989

50 Xmas lectures & masterclasses, ICMI Conf. on Popularization of Math, Leeds 1989.

51 New concept of stability in biology; Theoretical Biology Conf, Mexico, 1988.

52 Nonlinear dynamic systems, IMA Conference Cambridge 1989

53 Darwinian evolution. Catastrophes (ed J. Bourriau) CUP 1991