WORKING GROUP ON GEOMETRY

Recommended theorems in 3-dimensional geometry

E.C.Zeeman, August 2000.

Contents

Assumptions:

(a) Intersections

In general (i) 2 planes meet in a line
(ii) a line meets a plane in a point
(iii) 3 planes meet in a point.

Exceptions occur when (i) the 2 planes are parallel; (ii) the line is parallel to, or contained in, the plane; (iii) the 3 planes are parallel, or the line of intersection of 2 of them is parallel to, or contained in, the third.

(b) Two lines

2 lines are contained in a plane if and only if they meet or are parallel. If 2 lines are not contained in a plane then they neither meet nor are parallel, and they are called *skew*.

Definitions of perpendicular.

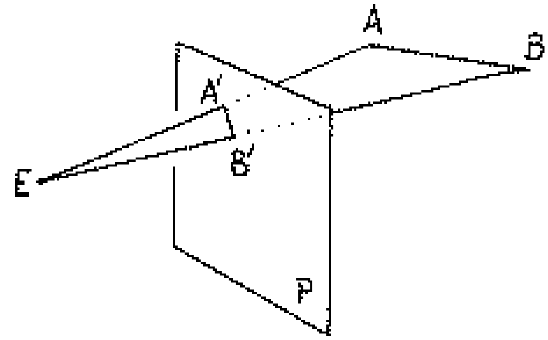
(i) 2 meeting lines are *perpendicular* if they are at right angles.

(ii) 2 skew lines are *perpendicular* if a line parallel to one and meeting the other is perpendicular to it.

(iii) A line is *perpendicular* to a plane if it is perpendicular to 2 non-parallel lines in the plane, and consequently to every line in the plane.

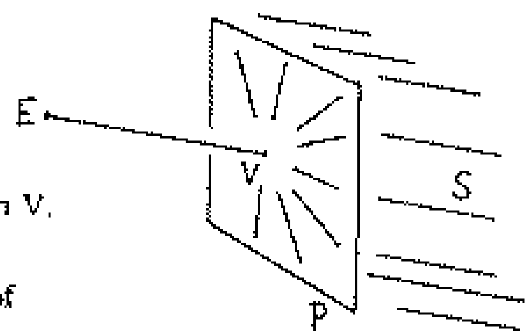
(iv) 2 planes are *perpendicular* if there is a line in one perpendicular to the other.

1. PERSPECTIVE



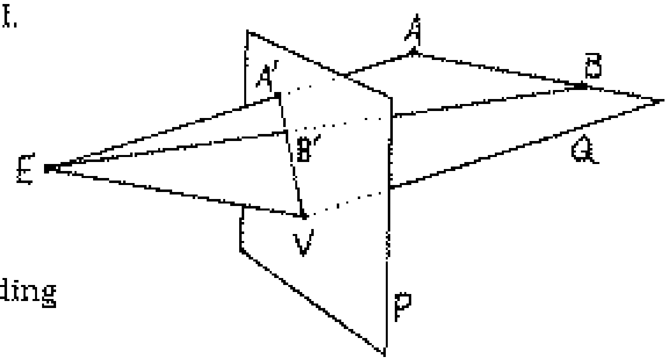
Imagine painting a 3-dimensional scene on a pane of glass P placed in between the the scene and the eye E.

Definition 1. The *image* A′ of a point A is where the ray EA pierces P. If also the image of B is B′ then the *image* of the line AB is A′B′.



Definition 2. The *vanishing point* V of a set S of parallel lines is where the parallel line through E pierces P.

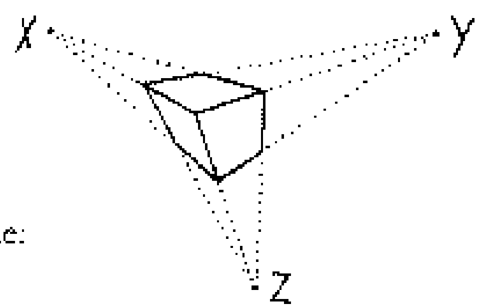
Theorem 1. All the images of S go through V.



Proof: It suffices to prove that the image of one line, when extended, goes through V, because by the same proof they all will. Let AB be the line. Then EV is parallel to AB by definition 2, and therefore they both lie in a plane Q. The 3 points A′,B′ &V all lie in both the planes P & Q, and hence on their line of intersection. Therefore extending A′B′ along this line goes through V.

Drawing a cube.

A cube has 3 sets of 4 parallel edges, and therefore a drawing of a cube needs 3 vanishing points. Choose an acute-angled triangle XYZ, and use the vertices as the vanishing points as shown.

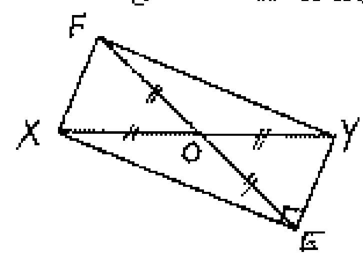


To then <u>see</u> the cube in perspective we must place the eye E in a position such that the lines EX,EY & EZ are parallel to the edges of the cube, which are perpendicular to each other. Therefore we define:

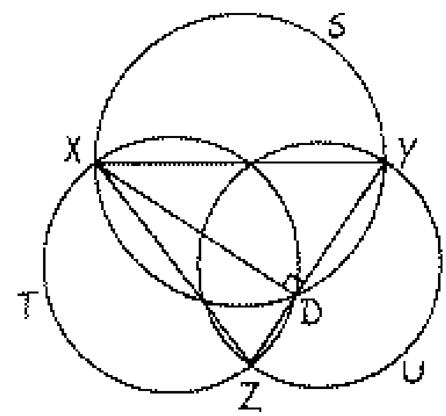
Definition 3. An *observation point* E is a point such that EX,EY & EZ are perpendicular to each other.

Theorem 2. There is exactly one observation point in front of P.

To prove the theorem we shall need the following lemma:
Lemma. If EX,EY are perpendicular then E lies on the sphere diameter XY.
Proof. Complete the rectangle XEYF by drawing lines through X,Y parallel to EY,EX to meet in F. Let O be the intersection of the diagonals XY & EF. Then by symmetry OX=OE=OY=OF. Therefore the circle centre O and radius OX is the circle diameter XY which goes through E. If we spin this circle about XY we obtain the sphere diameter XY.



Proof of Theorem 2. Let E be an observation point. Let S,T,U be the spheres diameters XY,XZ,YZ respectively. Since EX,EY are perpendicular E lies on S by the lemma, and similarly on T & U. Therefore we have to find the intersection of all 3 spheres. If C is the circle of intersection of S & T, then we have to find the intersection of C with the third sphere U. Now X lies on C. Let D be the foot of the altitude from X to YZ. Then D lies on S because $X\hat{D}Y$ is a right-angle. Similarly D lies on T and hence on C. Meanwhile D lies in between Y & Z because XYZ is an acute-angled triangle, and so D lies inside U. Meanwhile X lies outside U because $Y\hat{X}Z$ is less than a right-angle. Therefore C contains points both inside and outside U. Therefore C pierces U at 2 points. One of these points lies in front of P and the other is its mirror image behind P, because P is a plane of symmetry of all three spheres. Therefore there is exactly one observation point in front of P, as required.



Exercises.

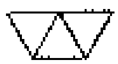
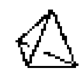
1. Prove the observation point lies in front of the orthocentre of XYZ.
2. Draw on the board an equilateral triangle XYZ of side 1 metre. Use the vertices as the 3 vanishing points to draw some rectangular boxes in perspective. View from $1/\sqrt{6}$ metre in front of the orthocentre and confirm that the boxes all look 3-dimensional and rectangular.
3. Prove that if XYZ is obtuse-angled then there is no observation point.

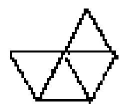
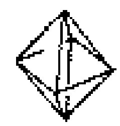
2 REGULAR SOLIDS

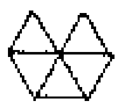
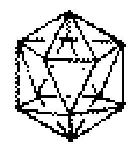
Definition. A *regular solid* has all its faces equal to the same regular polygon, and the same number of faces at each vertex.

Theorem. There are exactly 5 regular solids.

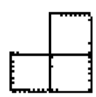
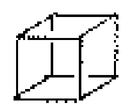
Proof. Given a regular solid, the ring of faces around a vertex contains at least 3 faces, and if the ring is cut open along an edge and flattened out it will occupy strictly less than 360 degrees. If the faces are equilateral triangles the ring can contain only 3,4 or 5 triangles because 6 would occupy the full 360 degrees: therefore there are 3 cases:

3 triangles  giving a tetrahedron 

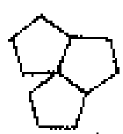
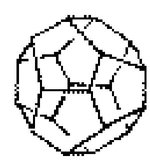
4 triangles  giving an octahedron 

5 triangles  giving an icosahedron 

If the faces are squares there is only 1 case because 4 squares would occupy 360 degrees:

3 squares  giving a cube 

If the faces are pentagons there is similarly only one case:

3 pentagons  giving a dodecahedron 

There are no more cases because 3 hexagons (or higher polygons) would occupy 360 degrees (or more).

Exercises.

1. Make the five regular solids.
2. Count the numbers of faces, edges & vertices for each solid.
3. Show that each solid obeys the formula: faces-edges +vertices = 2.
4. Explain in what way the cube & octahedron are dual to each other, and the dodecahedron & icosahedron are dual to each other, and the tetrahedron is self-dual.
5. Explain why the union of 2 tetrahedra glued together along a common face is not a regular solid. Show it obeys the formula above.
6. Make a *buckminster fullerine* (a carbon molecule or a football) out of 12 pentagons and 20 hexagons. Show it obeys the formula.

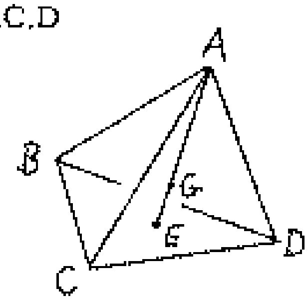
3 TETRAHEDRA

There are 4 theorems about 3 lines in a triangle meeting at a point: the 3 medians meet at the centroid. the 3 side-bisectors meet at the circumcentre, the 3 angle-bisectors meet at the incentre and the 3 altitudes meet at the orthocentre. We shall show that three of these theorems can be generalised to a tetrahedron in 3-dimensions, but the fourth cannot.

Definition 1. A *median* of a tetrahedron is the line joining a vertex to the centroid of the opposite face.

Theorem 1. The 4 medians of a tetrahedron are concurrent at a point G.

Proof. Let a,b,c,d be the vectors of the vertices A,B,C,D (with respect to some origin). Then the centroid E of BCD has the vector $e=(b+c+d)/3$. If G is the point with vector $g=(a+b+c+d)/4$ then $g=a/4 + 3e/4$. Therefore G lies on AE. Similarly for the other 3 medians.

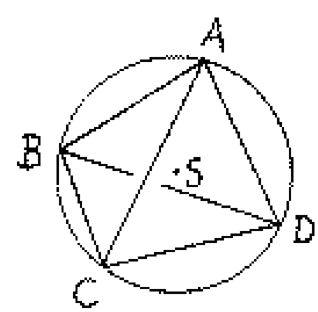


Exercise 1. Show that G is the midpoint of each of the 3 lines joining the midpoints of opposite edges of the tetrahedron.

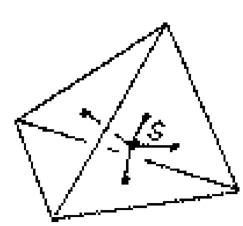
Definition 2. The *bisector* of a line AB is the plane perpendicular to, and through the midpoint of, AB: it is the set of points equidistant from A&B.

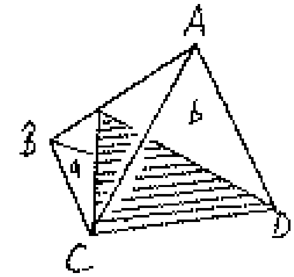
Theorem 2. The 6 edge-bisectors of a tetrahedron are concurrent at a point S, which is the centre of the circumsphere.

Proof. Let the tetrahedron be ABCD. Let S be the meet of the bisectors of AB,BC & CD.
Then AS=BS since S lies on the bisector of AB,
BS=CS since S lies on the bisector of BC, and
CS=DS since S lies on the bisector of CD.
Therefore S is equidistant from all 4 vertices, and so the sphere centre S through one vertex is the circumsphere going through all 4, and S lies on every edge-bisector.



Exercise 2. Show that the 4 lines through the 4 circumcentres of the 4 faces, and perpendicular to those faces, are concurrent at S.





Definition 3. Let a,b,c,d denote the faces of the tetrahedron opposite the vertices A,B,C,D.
The two faces a,b meet in the edge CD: define the *angle-bisector* of ab to be the plane through that edge making equal angles with a & b; it is the set of points equidistant from a & b.

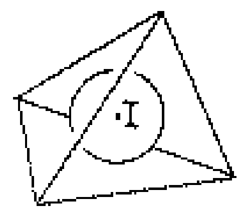
Theorem 3. The 6 angle-bisectors of a tetrahedron are concurrent at a point I, which is the centre of the insphere.

Proof. Let I be the meet of the angle-bisectors of ab,bc & cd. Then I is equidistant from a & b since it lies on the angle-bisector of ab,

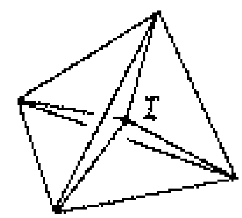
also from b & c since it lies on the angle-bisector of bc, and

also from c & d since it lies on the angle-bisector of cd.

Therefore I is equidistant from all 4 faces, and so the sphere centre I touching one face is the insphere touching all 4, and I lies on every angle-bisector.



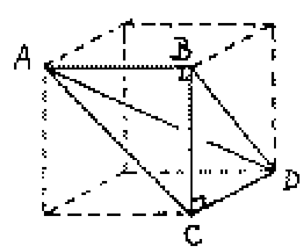
Exercise 3. Show that the 4 lines going through the 4 vertices, each equidistant from the 3 faces at that vertex, are concurrent at I.



Definition 4. An *altitude* of a tetrahedron is a line through a vertex perpendicular to the opposite face.

Theorem 4. In general the 4 altitudes of a tetrahedron are <u>not</u> concurrent.

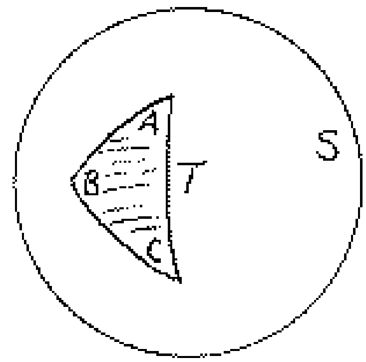
Proof. We construct a counterexample. Let ABCD be the tetrahedron inscribed in a cube as shown. Then the altitudes through A & D are AB & DC, which do not meet.



Exercise 4. Show that if the opposite edges of a tetrahedron are perpendicular then the foot of each altitude is the orthocentre of the opposite face, and the 4 altitudes are concurrent. Give two examples of such tetrahedra.

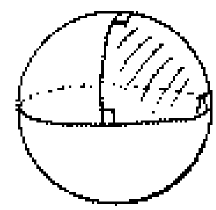
4. SPHERICAL TRIANGLES

The theorem about the 3 angles of a triangle adding up to 180 degrees can be generalised to spherical triangles, and then used to give the sum of the 4 solid-angles of a tetrahedron.



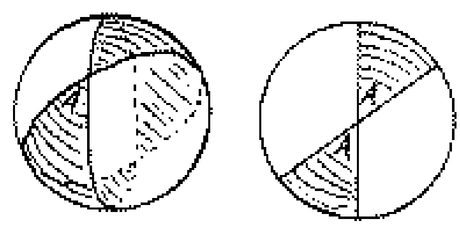
Definition 1. A *great circle* on a sphere is the intersection of the sphere with a plane through its centre. A *spherical triangle* consists of 3 arcs of 3 great circles. Let A,B,C be the angles at the vertices (or more precisely between the tangents to the sides at each vertex). Let S = surface area of the sphere
T = surface area of the triangle.

Theorem 1. A+B+C = 180(1+4T/S).

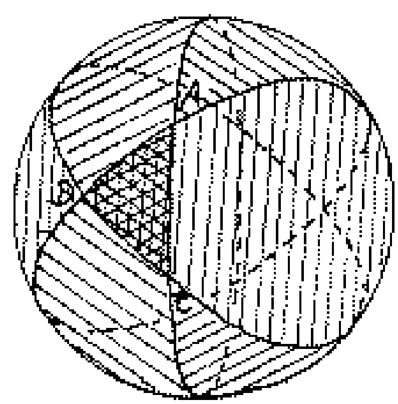


Example 1. The triangle shown has 3 right-angles and so A+B+C=270. Meanwhile T occupies a quarter of the northern hemisphere and so T/S=1/8.
Example 2. If T becomes very small compared with S (like a triangle on the surface of the earth) then the sum of the angles tends to 180.

To prove the theorem we need the following lemma.

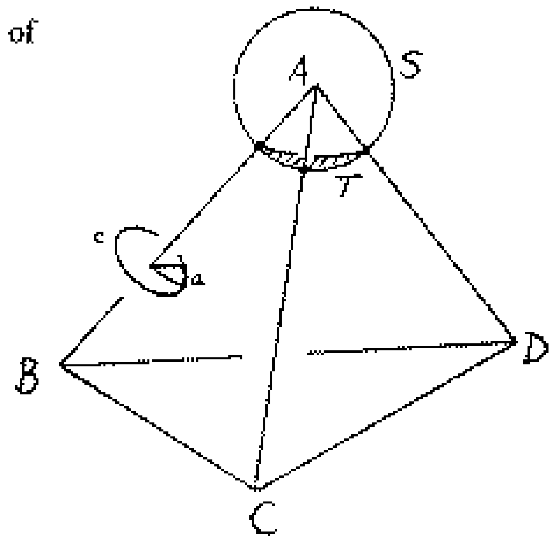


Definition 2. Define the *A-lune* to be the area between the 2 great circles through A.
Lemma. A-lune/S = A/180.
Proof. Looking down on S from above A
A-lune/S = 2A/360 = A/180.
Proof of Theorem 1. The 3 lunes cover the whole sphere, but cover the triangle 3 times, which is 2 times too many, and the same with the antipodal triangle. Therefore
A-lune + B-lune + C-lune = S + 4T.
Therefore
(A-lune + B-lune + C-lune)/S = 1 + 4T/S.
Therefore by the lemma
(A+B+C)/180 = 1 + 4T/S.
Multiplying by 180 gives the theorem.



Definition 3. In a tetrahedron ABCD define the *solid-angle* A to be T/S, where S is the area of a small sphere centre A, and T is the area of the triangle cut off by the tetrahedron.

Definition 4. Given an edge AB, define the *dihedral-angle* of AB to be a/c, where c is the length of the circumference of a small disc centred on and perpendicular to AB, and a is the length of the arc cut off by the tetrahedron. Notice that 1 unit of dihedral angle equals 360 degrees.



Theorem 2. In a tetrahedron

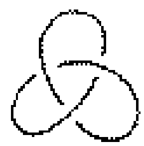
(the sum of the 4 solid-angles) = (the sum of the 6 dihedral-angles) - 1.

Exercises.

1. Deduce Theorem 2 from Theorem 1.
2. Show that in a regular tetrahedron: dihedral-angles = $\cos^{-1}(1/3)$,
   solid-angles = $(3/2)\cos^{-1}(1/3) - 1/4$.
3. Calculate the dihedral and solid-angles of the tetrahedron used in the proof of Theorem 4 in Section 3.

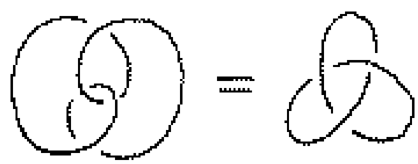
5 KNOTS AND LINKS

Topology is sometimes called "rubber" geometry because it studies properties like knotting and linking, that are much deeper than those in the previous sections because they persist under more general rubber-like transformations. Consequently the style of proof will be quite different.

knotting  linking 

Definition 1. A *knot* is a closed curve in 3 dimensions. Two knots are *equal* if one can be moved into the other.

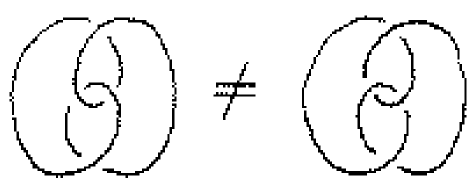
Example 1.

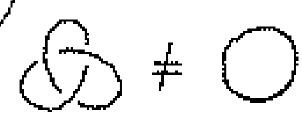


Trefoil

Proof.

Definition 2. Two knots are *unequal* if one cannot be moved into the other.

Trefoil  Square knot

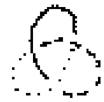
A curve is *knotted* if it is unequal to a circle. 

To prove inequality (or knottedness) we need to introduce an *invariant*, namely a property of a knot that does not vary if we move the knot, and show that the two knots have different values of the invariant.

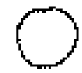
Let K be a picture of a knot, with a finite number of crossings. At each crossing the *underpass* is indicated by a break in the curve, and so the curve is broken into a finite number of arcs.

Definition 3. We say K has *code* 3 if it can be 3-coloured as follows. Each arc is one colour, and (1) at least 2 colours are used

(2) at each crossing 1 or 3 colours are used (for the overpass & the 2 sides of the underpass).

Lemma 1. The trefoil has code 3. **Proof.** 

Lemma 2. The circle has not code 3.

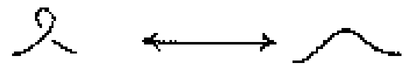
Proof. Otherwise being all one colour would violate condition (1). 

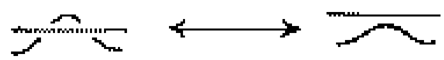
Theorem 1. Code 3 is an invariant.

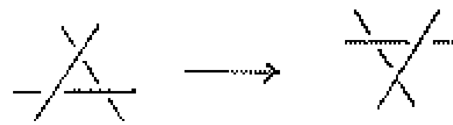
Corollary. The trefoil is knotted.

Proof of the Theorem.

We have to show that if K can be 3-coloured, and K is moved to L, then L can be 3-coloured. Consider the following 3 types of elementary move:

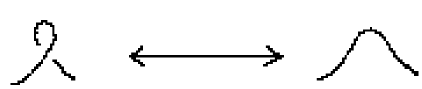
Type I (& its inverse) 

Type II (& its inverse) 

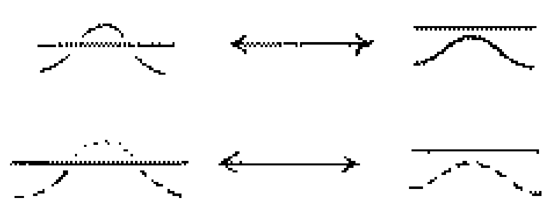
Type III (which equals its own inverse by turning upside down) 

If K→L is a long complicated move imagine taking a film of it and examining the film frame by frame. At each frame there is either no change in the configuration of arcs from the previous frame, or else there has been an elementary move. Therefore we can interpret the complicated move K→L as a finite sequence of elementary moves. For instance in the proof of Example 1 above the first and last steps represent no change in the configuration, while the second and third steps are elementary moves of types III and I.

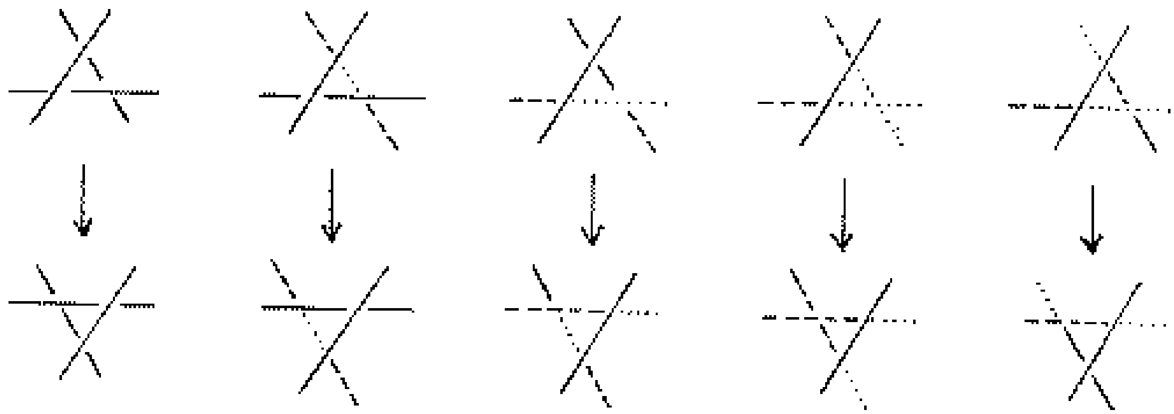
If we prove the theorem for elementary moves then it follows for any sequence of such, and hence for any move. In each case we are given a 3-colouring before the elementary move, and have to show there is a 3-colouring after the the elementary move, without changing the colouring of the rest of the knot, or of the ends of the elementary move that are attached to the rest of the knot.

Type I (& its inverse) 

Type II (& its inverse): there are 2 cases depending on whether the ends are coloured the same or different



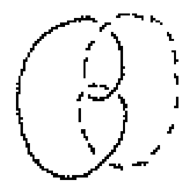
Type III: there are 5 cases, and in each case we have to show that it is possible to achieve a colouring satisfying condition (2) by recolouring the little arc without changing any of the other arcs, since they are attached to the rest of the knot.



This completes the proof of Theorem 1.

Lemma 3. The square-knot has not code 3.

Proof. The square knot contains 4 arcs, and so 2 of them must be the same colour. But any 2 arcs meet at some crossing. Therefore the third arc at this crossing must be the same colour by condition (2). Similarly the fourth arc must also be the same colour, violating condition (1).

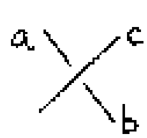


Corollary. Trefoil $\neq$ square-knot.

However, this invariant is no good for proving that the square-knot is knotted because neither the square-knot nor the circle has code 3. Therefore we need to generalise the invariant, and for this we shall use arithmetic modulo p as follows.

Definition 4: Mod p arithmetic. Let p be an odd prime. The set of *integers mod p* is the set of integers 0,1,2,...,p-1. Given two integers a,b we write a≡b(mod p) if they differ by a multiple of p.

Definition 5. We say K has *code* p if the arcs can be labelled with integers mod p such that (1) at least two arcs are labelled differently, and



(2) at each crossing the average of the two sides of the underpass equals the overpass (mod p):

$$a+b=2c \pmod{p}.$$

Lemma 4. The square-knot has code 5.

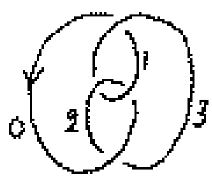
Proof. Going round the knot we check each crossing:

0+2=2

2+3=0(mod 5)

3+1=4

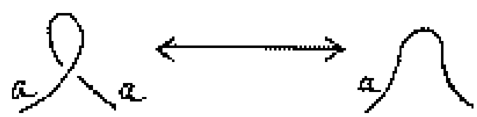
1+0=6(mod 5)



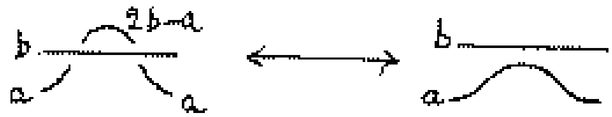
Theorem 2. Codes are invariant.

Proof. It suffices to check the elementary moves.

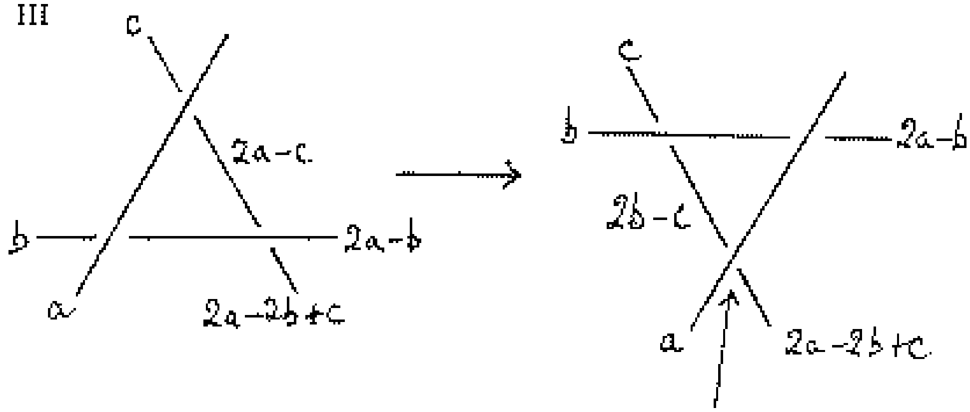
Type I (& its inverse)



Type II (& its inverse)



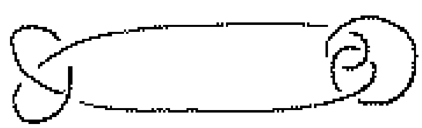
Type III



Check: (2b-c)+(2a-2b+c) = 2a.

This completes the proof of Theorem 2.

Definition 6. The *product* of 2 knots is given by joining them together.

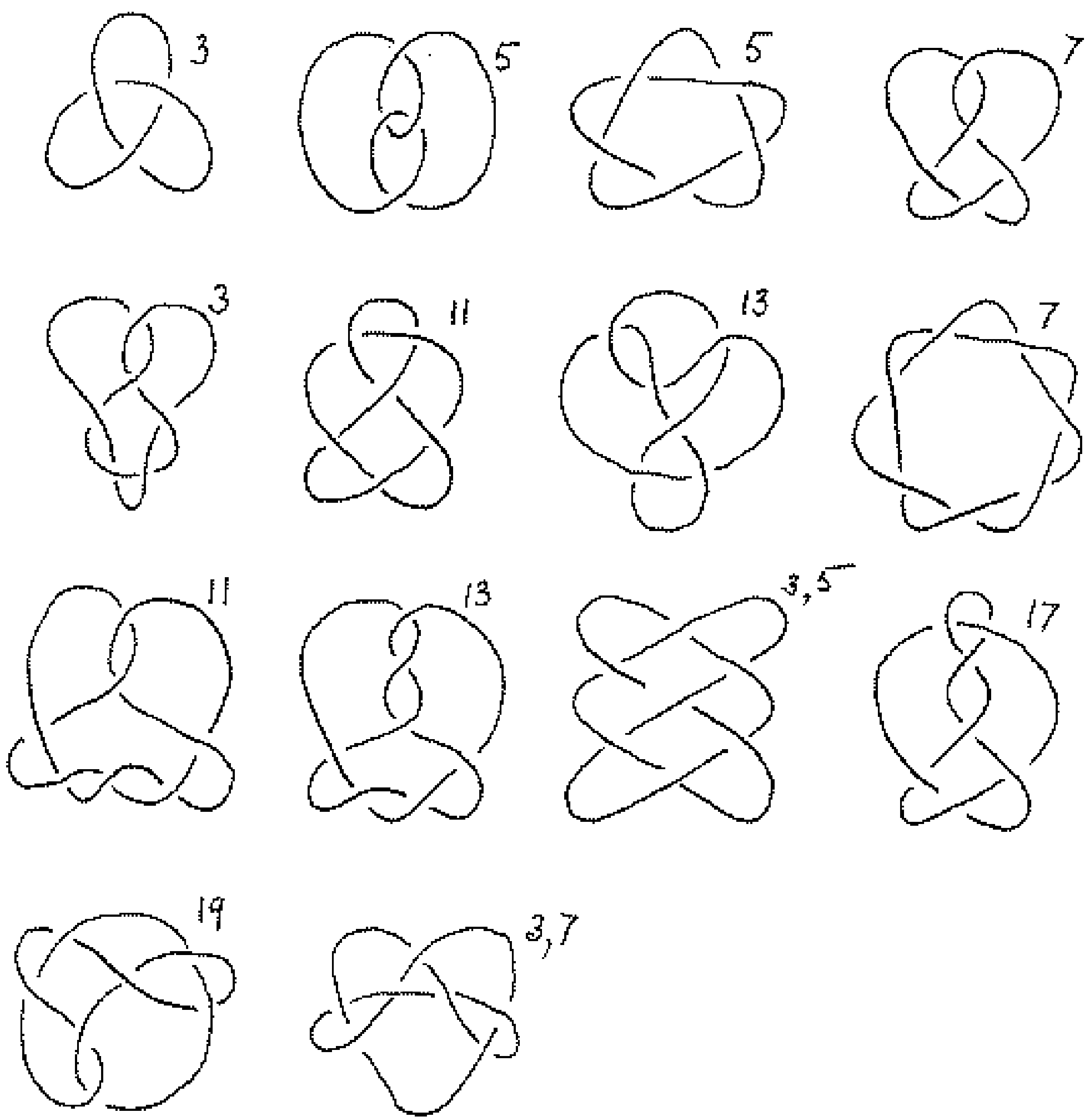
trefoil  square-knot

A knot is called *prime* if it is not the product of two simpler knots. The list of all 14 prime knots with less than 8 crossings is shown below.

Exercises.

1. Show that when p=3 then Definition 5 is equivalent to Definition 3.
2. Show that the product of the trefoil and square knot has codes 3&5.

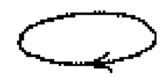
3. Calculate the codes of the prime knots with less than 8 crossings as shown below. Since the circle has no codes this proves that they are all knotted. It does not prove. however, that those with the same code are unequal, and this requires a more sophisticated invariant.

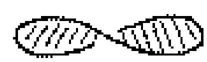


Linking is one of the most characteristic features of 3-dimensions. It is intuitively and experimentally obvious that linked curves cannot be separated, but we shall prove this mathematically by constructing an invariant called the *linking number* L that measures how many times one curve links the other.

(Incidentally the same proof can be used to show that two spheres can be linked in 5-dimensions, where intuition is less obvious and experiment is impossible.)

Definitions. To *orient* a curve means to choose one or other of the two directions going round the curve; the orientation is indicated by an arrow. To *span* a curve means to choose a disc whose boundary is the curve. (The disc may itself be curved, and is allowed to intersect itself if the curve happens to be knotted.)





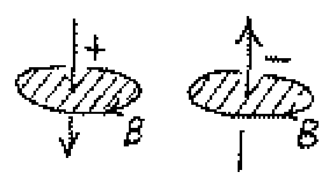
Definition of linking number L. Given two curves A,B we make 3 choices:

(1) orientations of A & B;

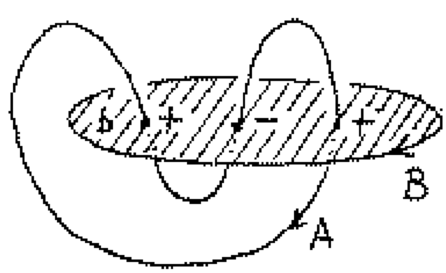
(2) either A or B to span, say B; and

(3) a disc b spanning B.

Then A will pierce b in a finite number of points. We call a particular piercing *positive* if A pierces b in the direction that a right-handed corkscrew would move if it were screwed in the direction of the orientation of B; otherwise call it *negative*. Let P be the number of positive piercings and N the number of negative piercings. Define the *linking number* L to be the difference between P&N.



Example.



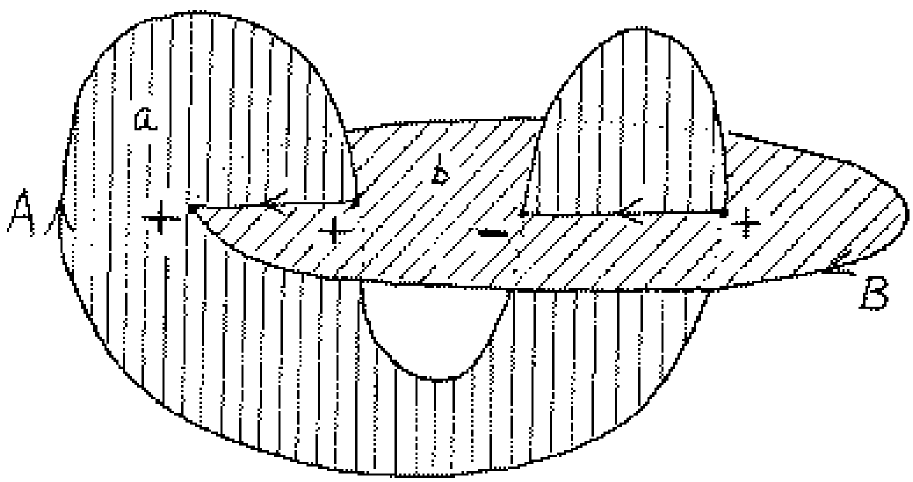
P=2
N=1
L=1

Theorem. L is invariant.

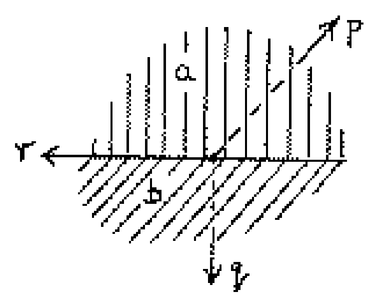
Proof. We have to prove firstly that L is independent of the 3 choices, and secondly that it does not vary when the curves are moved. The second part is easy because if the disc is moved along with the curves then the number of piercings will be conserved. Hence the burden of proof lies in showing that L is independent of the 3 choices.

(1) If one of the orientations is reversed then the sign of each piercing is reversed. Therefore P & N are interchanged, and their difference L is the same.

(2) Suppose we chose to span A rather than B, and chose a disc a spanning A. Let P' & N' be the numbers of positive & negative piercings of a by B, and let L' be their difference. We have to show that L=L'.



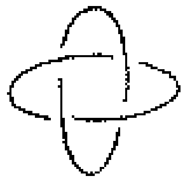
The intersection of a & b consists of a finite number of arcs and closed curves. Forget the closed curves and concentrate on the arcs because their ends will be the piercings of b by A and a by B. Orient the arcs so that at each point, if r is a vector giving the direction of orientation, and p,q are vectors giving positive piercings of a,b, then {p,q,r} is a right-handed set of axes. Then the



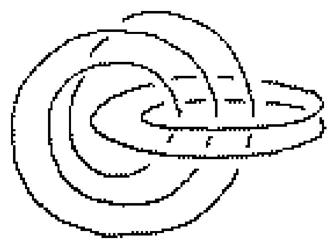
front ends of the arcs will be the positive piercings of a by B and the negative piercings of b by A, while the back ends will be the complementary piercings. But the number of front ends is the same as the number of back ends. Therefore P'+N=N'+P. Therefore N-P=N'-P'. Therefore L=L', as required.

(3) Finally suppose we chose a different disc b" spanning B, giving rise to a linking number L". Then L"=L'=L by (2) above, and so L"=L as reqired. This completes the proof of the theorem.

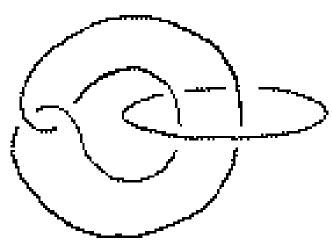
Exercises.

1. Show that  has L=2. Is it equal to  ?

2. Calculate the linking numbers of



3. Show that the link below has L=0. This does not imply, however, that the curves are unlinked. To prove that they are in fact linked show that unlinked curves have code 3, but this link does not.



4. Draw an example of 3 linked curves that are pairwise unlinked.

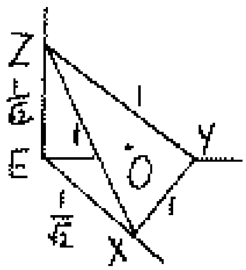
6. SOLUTIONS TO THE EXERCISES

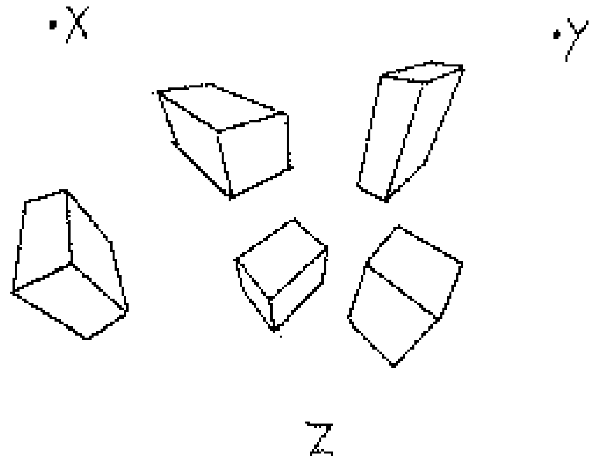
Section 1.

1. Let E be the observation point, and O the orthocentre of XYZ. The plane containing the circle C is perpendicular to P and contains E & the altitude XD, and hence O and the line OE. Similarly the planes containing the other 2 circles of intersections of the 3 spheres are perpendicular to P and contain OE. Therefore OE is perpendicular to P, as required.

2. With respect to axes EX,EY,EZ the orthocentre O has coordinates $(1/3\sqrt{2}, 1/3\sqrt{2}, 1/3\sqrt{2})$. Therefore

$$EO = \sqrt{3/18} = 1/\sqrt{6}.$$





3. If X is obtuse then X lies inside the sphere U, along with D, and so C lies inside U. Therefore C does not meet U. Therefore the 3 spheres do not meet, and so there is no observation point. If Y or Z is obtuse then D lies outside U, along with X, and so C lies outside U. Again C does not meet U, and so there is no observation point.

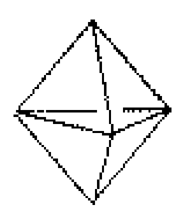
Section 2.

2&3.

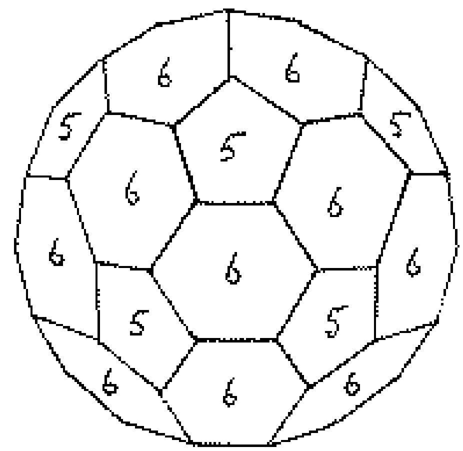
Solid	faces	edges	vertices	
tetrahedron	4	6	4	4-6+4=2
cube	6	12	8	6-12+8=2
octahedron	8	12	6	8-12+6=2
dodecahedron	12	30	20	12-30+20=2
icosahedron	20	30	12	20-30+12=2

4. Put a dual-vertex at the centre of each face. Join 2 dual-vertices with a dual-edge if the corresponding faces have an edge in common. Then below each vertex there is a dual-face, and they bound the dual-solid.

5. It does not have the same number of faces at each vertex, 3 faces at 2 vertices and 4 faces at 3 vertices. Faces=6, edges=9, vertices=5, and 6-9+5=2.



6. The buckminster fullerine.
   Faces = 12+20 = 32,
   edges = $(12\times5 + 20\times6)/2 = 90$,
   vertices = $(12\times5 + 20\times6)/3 = 60$.
   Therefore $32-90+60=2$.

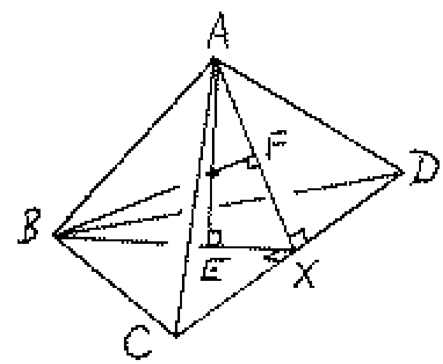


Section 3.

1. Let X,Y be the midpoints of AB,CD. Then $x=(a+b)/2$, $y=(c+d)/2$ and so $g=(a+b+c+d)/4 = (x+y)/2$. Therefore G is the midpoint of XY.

2. The line perpendicular to ABC through the circumcentre of ABC is the set of points equidistant from A,B,C, and therefore contains S. Similarly for the other 3 lines.

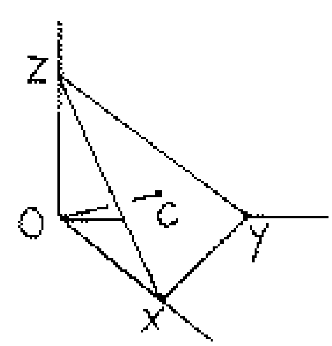
3. The line through A equidistant from b,c,d goes through I, and similarly for the other 3 lines.



4. Let AE be an altitude of the tetrahedron, and suppose BE meets CD in X. Now AE is perpendicular to BCD, and so AE is perpendicular to CD. Meanwhile AB is perpendicular to CD by hypothesis. Therefore ABE is perpendicular to CD. Therefore BX is perpendicular to CD, and is hence an altitude of BCD. Therefore E lies on all the altitudes of BCD, and is hence the orthocentre of BCD.

Meanwhile AX is perpendicular to CD, and is hence an altitude of ACD, containing the orthocentre F of ACD. Therefore the altitude BF of the tetrahedron lies in the plane ABE, and hence meets AE. Therefore all 4 altitudes of the tetrahedron meet pairwise, and are not coplanar, and so they must be concurrent.

Examples (i) The regular tetrahedron.
(ii) The tetrahedron OXYZ where X,Y,Z are the unit points on the axes OX,OY,OZ. Let C be the centroid of XYZ. Then the altitudes of the tetrahedron are OC, XO,YO,ZO, which are concurrent at O.



Section 4.

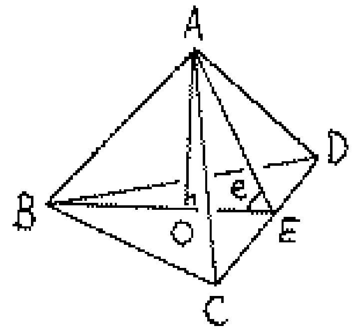
1. Let d(AB)=dihedral-angle of AB, and s(A)=solid-angle of A. Then
d(AB)+d(AC)+d(AD)=(1+4s(A))/2, by Theorem 1 (since 180 degrees equals half a dihedral unit).

Summing over the 4 vertices repeats each dihedral-angle twice:

2(sum of the 6 dihedral-angles) = 2 + 2(sum of the 4 solid-angles).

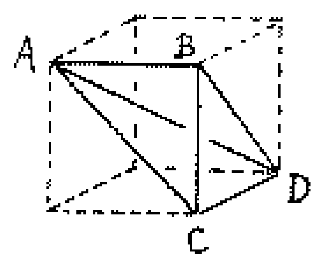
Dividing by 2 gives Theorem 2.

2. Let e be the dihedral-angle of a regular tetrahedron ABCD. Let E be the midpoint of CD, and O the centroid of BCD. Then 3(OE)=BE=AE. Therefore cos e=1/3. Therefore $e=\cos^{-1}(1/3)$. The solid-angle = (6e-1)/4 = $(3/2)\cos^{-1}(1/3)-1/4$.



3. Dihedral-angles AB,CD = 1/8
AD = 1/6
AC,BC,BD = 1/4
Solid-angles A,D = 1/48
B,C = 1/16.

Check: 2/48 + 2/16 = 2/8 + 1/6 + 3/4 - 1.

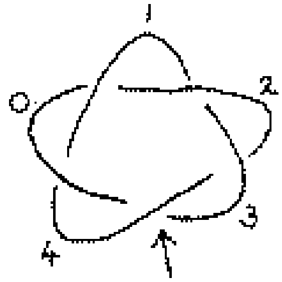


Section 5: knots.

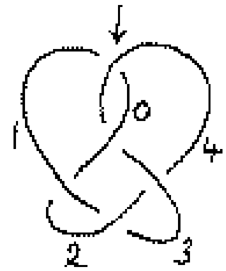
1. Use as colours 0,1,2. If one colour is used at a crossing then trivially the overpass is the average of the underpasses; if 3 colours are used then 0+1=4(mod 3), 1+2=0(mod 3), and 2+0=2.

2. The product has code 3 by labelling the trefoil appropriately with the integers mod 3 and labelling the square-knot all the same. Similarly it has code 5 by labelling the square-knot appropriately with the integers mod 5 and labelling the trefoil all the same.

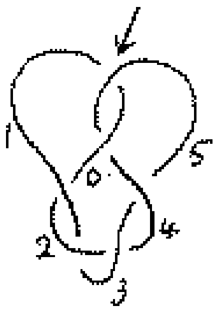
3. The first two cases of trefoil and square-knot have already been done. In each of the other dozen cases we start by labelling one crossing with 0,1,2, then the next with 1,2,3, and so on preserving averages until the penultimate crossing (indicated by an arrow) which gives an equation for p. Then the last crossing is satisfied automatically, and provides a convenient check for the computation.



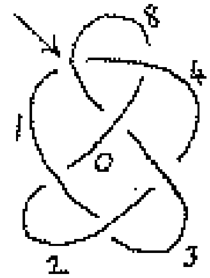
$3+0=8 \pmod p$
$5=0 \pmod p$
$p=5$



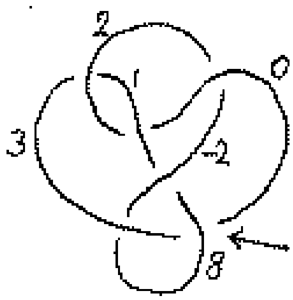
$0+1=8 \pmod p$
$p=7$



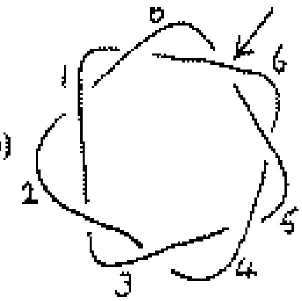
$0+1=10 \pmod p$
$9=0 \pmod p$
$p=3$



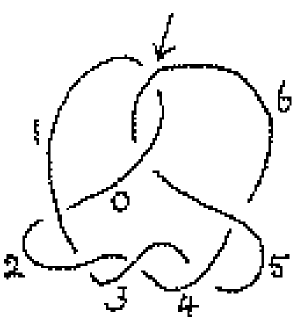
$4+1=16 \pmod p$
$p=11$



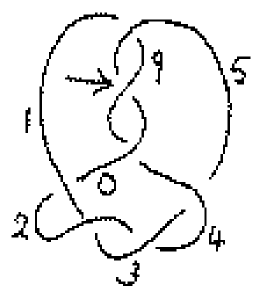
$3+0=16 \pmod p$
$p=13$



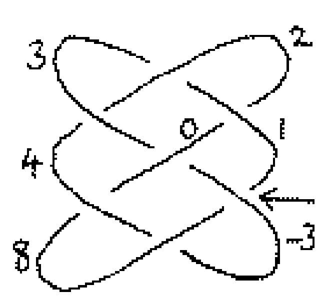
$5+0=12 \pmod p$
$p=7$



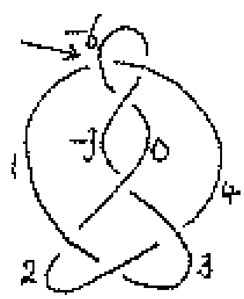
$0+1=12 \pmod p$
$p=11$



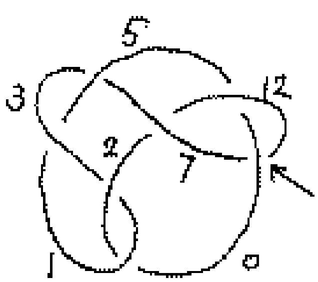
$0+5=18 \pmod p$
$p=13$



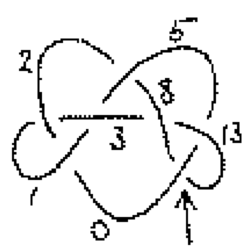
$8+1=-6 \pmod p$
$15=0 \pmod p$
$p=3$ or 5



$1+4=-12 \pmod p$
$p=17$



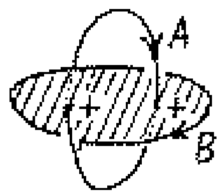
$7+12=0 \pmod p$
$p=19$



$8+13=0 \pmod p$
$21=0 \pmod p$
$p=3$ or 7

Section 5: links.

1.



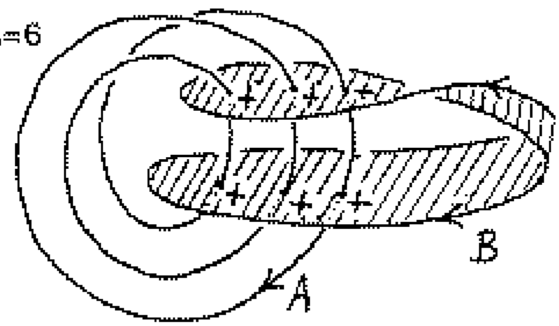
L=2.

Yes, equal. **Proof:**

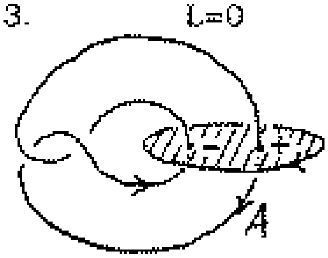
2.

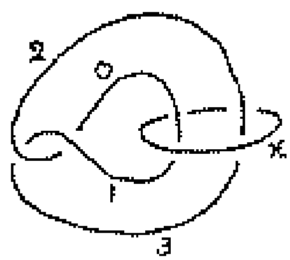
L=3

L=6



3. L=0





Suppose it had code p.
Then 0+1=2x=2+3(mod p)
Therefore 4=0(mod p),
contradicting p odd.
Therefore it has no codes.

4. The **Borromean link.** It can be seen that any 2 of the 3 curves are unlinked. The 3 together are linked, however, because they cannot be moved apart. **Proof**. 3 unlinked curves have all codes (because there are no crossings), but the Borromean link has no codes. For suppose it had code p. and was labelled as shown. Then a+a'=2b(mod p), and a+a'=2b'(mod p). Therefore 2b=2b'(mod p), and so b=b'(mod p) since p is odd. Similarly a=a'(mod p). Therefore 2a=2b(mod p). and so a=b(mod p). Similarly b=c(mod p), violating condition (1). and giving a contradiction.