A NEW METHOD FOR SOLVING PARTIAL AND ORDINARY DIFFERENTIAL EQUATIONS USING FINITE ELEMENT TECHNIQUE

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ABSTRACT

In this paper we introduce a new method for solving partial and ordinary differential equations with large first, second and third derivatives of the solution in some part of the domain using the finite element technique (here called the Galerkin-Gokhman method). The method is based on the application of the Galerkin method to a modified differential equation. The exact solution of the modified equation is the Galerkin approximation for the unknown function with exact values of the unknown at the nodal points.

An application of the Galerkin-Gokhman method to a general second order nonlinear ordinary differential equation and to Navier-Stokes equations in the case of developing flow in a pipe is formulated. We also include the results of an application of the Galerkin-Gokhman method to two specific ordinary differential equations. One is: $y - dy/dx = 0$, the other one is a second order nonlinear equation describing fully developed turbulent flow in a pipe.

NOMENCLATURE

G_i	global shape function for Galerkin approximation
l	Prandtl mixing length
p	pressure
R, D	radius and diameter of pipe
r, z	cylindrical coordinates
Re	Reynolds number for flow through the pipe ($\bar{v}_z D/\nu$)
v_r, v_z	velocity components in r and z directions
$\bar{v}_z$	average axial velocity in a pipe
V_0	functional space
x	independent variable
y	solution of the differential equation
y_s	spline representation of the solution
$\tilde{y}$	Galerkin approximation for the solution
y_e	exact solution of the differential equation
$(\delta v_z)_{\max}$	maximum error in computation of v_z
Δy	difference between y_s and $\tilde{y}$
η	relative radius ($= r/R$)
λ	unitless parameter in the formula for G_i

ν	Newtonian kinematic viscosity
ν_t, ν_e	Boussinesq turbulent and effective kinematic viscosities
ρ	density of fluid
τ	tangential stress in the fluid

INTRODUCTION

The Galerkin method is widely used for finite element solutions of Navier-Stokes equations. It provides acceptable accuracy of the solution for the flows with small values of Reynolds number [1,2]. However, when first, second and third derivatives of the solution are large in some part of the domain the application of the Galerkin method to ordinary and partial differential equations leads to unacceptable errors. We shall refer to functions with large first, second and third derivatives in some part of the domain as steep. The most important example of this phenomenon occurs in Navier-Stokes equations for internal flow at high Reynolds numbers. In this case, the velocity of the flow changes in a highly nonlinear fashion at the wall of the conduit which causes significant errors in the solution [3,4].

In order to provide a solution of Navier-Stokes equations for internal flows at high Reynolds numbers the author of the present paper in 1986-1987 developed an extension of the Galerkin method which further will be called the Galerkin-Gokhman method. In 1988-1989 the Electric Power Research Institute, Palo Alto, funded a research project "Applicability of the Galerkin-Gokhman Method to the Solution of Navier-Stokes Equations Using Finite Element Technique". The results of this research showed the superiority of the Galerkin-Gokhman Method and its capability to yield an accurate solution.

The present paper is the first in a series of papers devoted by author to the Galerkin-Gokhman method and the application of this method to the solution of Navier-Stokes equations.

1. FORMULATION OF THE GALERKIN METHOD FOR THE NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

We will demonstrate the Galerkin-Gokhman method as applied to an ordinary differential equation (ODE). Since the Navier-Stokes equations are second order partial differential equations (PDE) with second order nonlinearity we will consider a second order ODE with second order nonlinearity:

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0 \qquad x \in (0,1) \tag{1}$$

with boundary conditions:

$$y(0) = 0,\ y'(1) = 0. \tag{2}$$

A solution y is sought in a functional space V_0 of sufficiently smooth functions satisfying a homogeneous condition at $x = 0$.

We assume that the space V_0 has a countable basis $G_1, G_2, ...$, which means that any function $w \in V_0$ can be expressed as an infinite combination of basis functions [5]:

$$w = \sum_{i=1}^{\infty} a_i G_i. \tag{3}$$

Now we try to an select element $y \in V_0$ for which the left part of (1) is identically zero. The left part of (1) is identically zero if its projection on each basis function is zero:

$$\int_0^1 f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right) G_i dx = 0. \tag{4}$$
$$i = 1, 2, ...$$

The idea behind the Galerkin method consists of taking the finite dimensional subspace V_{0N} of V_0 spanned by basis functions $G_1, G_2, .., G_N$.

An approximate problem then is defined as:

Find $\tilde{y} \in V_{0N}$ such that

$$\int_0^1 f\left(\frac{d^2\tilde{y}}{dx^2}, \frac{d\tilde{y}}{dx}, \tilde{y}, x\right) G_i dx = 0. \tag{5}$$
$$i = 1, 2, ...n$$

The approximate solution, being a function in V_{0N}, has the form:

$$\tilde{y} = \sum_{i=1}^{n} a_i G_i. \tag{6}$$

The most convenient presentation for $\tilde{y}$ is:

$$\tilde{y} = \sum_{i=1}^{n} y_i G_i \tag{7}$$

where y_i are unknown values of the function $y = y(x)$ at the points x_i $(i = 1, 2, ..., n)$.

The easiest way to define the continuous functions $G_i(x)$ is to require that:

$$G_i(x_i) = 1$$

$$G_i(x) = 0 \quad \text{for} \quad x \leq x_{i-1} \qquad\qquad x \geq x_{i+1}$$

Defined this way basis functions are called global shape functions. Now we can write:

$$\frac{d\tilde{y}}{dx} = \sum_{i-1}^{n} y_i \frac{dG_i}{dx} \tag{8}$$

and

$$\frac{d^2\tilde{y}}{dx^2} = \sum_{i=1}^{n} y_i \frac{d^2G_i}{dx^2}. \tag{9}$$

It is clear from (9) that in order to solve a second order equation the shape functions have to be at least parabolic (for the linear shape functions the terms with second derivatives are identically zero). All of the domain of the variable x (0,1) where we are looking for solutions must be divided into $(n-1)/2$ elements (n has to be the odd number).

For a general element $k = (i+1)/2$ $(i = 1, 3, ..., n-2)$ with vertex nodal points x_i and x_{i+2} and middle nodal point $x_{i+1} = 0.5(x_i + x_{i+2})$, the parabolic shape functions are described bellow.

At the point x_i:

$$G_i(x) = 0 \qquad [x \le x_{i-2} \text{ and } x \ge x_{i+2}], \tag{10}$$

$$G_i(x) = 0.5\lambda(\lambda + 1) \qquad [x_{i-2} \le x \le x_i], \tag{11}$$

where

$$\lambda = 2(x - x_{i-1})/(x_i - x_{i-2}) \qquad [-1 \le \lambda \le 1],$$

$$G_i(x) = 0.5\lambda(\lambda - 1) \qquad [x_i \le x \le x_{i+2}] \tag{12}$$

where

$$\lambda = 2(x - x_{i+1})/(x_{i+2} - x_i) \qquad [-1 \le \lambda \le 1].$$

At the point x_{i+1}:

$$G_{i+1}(x) = 0 \qquad [x \le x_i \text{ and } x \ge x_{i+2}], \tag{13}$$

$$G_{i+1}(x) = 1 - \lambda^2 \qquad [x_i \le x \le x_{i+2}] \tag{14}$$

where

$$\lambda = 2(x - x_{i+1})/(x_{i+2} - x_i) \qquad [-1 \le \lambda \le 1].$$

Now substituting (7), (8) and (9) into (5) we obtain a system of $n-1$ algebraic equations with $n-1$ unknowns. We do not write the equation of type (5) for $i = 1$, because y_1 is known from the boundary condition $y_1 = y(0) = 0$.

The system of equations (5) is a system of quadratic equations, since our ODE is of second order of nonlinearity. The system of second order algebraic equations can be iteratively solved using the Newton-Raphson method.

Naturally the approximate solution differs from the exact solution of the equation (1)

$$y_i \neq y_e(x_i) \tag{15}$$

where $y_e(x_i)$ is the exact value of the function at $x = x_i$ and y_i is the result of the Galerkin solution.

The idea of the Galerkin-Gokhman method is based on the modification of the equation (1) in such a way that it will yield a solution in the form of a finite linear combination of the basis functions, i.e. of the form (7). Let us assume that the exact solution of the equation (1) $y_e = y_e(x)$ is known and let

$$\Delta y = y_e(x) - \sum_{i=1}^{n} y_e(x_i) G_i \qquad x \in (0,1) \tag{16}$$

then

$$\frac{d(\Delta y)}{dx} = \frac{dy_e(x)}{dx} - \sum_{i=1}^{n} y_e(x_i) \frac{dG_i}{dx} \tag{17}$$

and

$$\frac{d^2(\Delta y)}{dx^2} = \frac{d^2 y_e(x)}{dx^2} - \sum_{i=1}^{n} y_e(x_i) \frac{d^2 G_i}{dx^2}. \tag{18}$$

If we use in the equation (1) instead of y the corrected value $y + \Delta y$, then the equation (1) becomes:

$$f\left\{\left[\frac{d^2 y}{dx^2} + \frac{d^2(\Delta y)}{dx^2}\right], \left[\frac{dy}{dx} + \frac{d(\Delta y)}{dx}\right], [y + \Delta y], x\right\} = 0, \tag{19}$$

i.e. the modified equation:

$$f_m\left(\frac{d^2 y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0. \tag{20}$$

It follows from (16), (17) and (18), that the substitution of $\tilde{y}_e$ into (19) leads to

$$f\left(\frac{d^2 y_e}{dx^2}, \frac{dy_e}{dx}, y_e, x\right) = 0.$$

Therefore, the solution of (20) is:

$$\tilde{y}_e = \sum_{i=1}^{n} y_e(x_i) G_i \qquad\qquad x \in (0,1).$$

Thus applying the Galerkin method to the modified equation (20) we obtain a system of quadratic algebraic equations:

$$\int_0^1 f_m\left(\frac{d^2 \tilde{y}}{dx^2}, \frac{d\tilde{y}}{dx}, \tilde{y}, x\right) G_i(r) = 0 \tag{21}$$

$$i = 2, ..., n$$

and since $\tilde{y}_e$ is the exact solution of the modified equation (20) it is also the solution of the system of equations (21). Consequently if one finds the exact corrective function Δy using the method of iteration, the system (21) will yield the exact values of $y(x_i)$.

With respect to this it is important to emphasize that in order to be a solution of (20) the function $\tilde{y}$ must have continuous first and second derivatives. Therefore the parabolic shape functions in principle are not providing the exact solution when one uses the Galerkin-Gokhman method. The alternative shape functions are third order polynomials specified for two nodal point elements (see section 2). These shape functions yield $\tilde{y} \in C^2$. However, practical computations show that the utilization of parabolic functions provides sufficient accuracy if the segments $\Delta x_i = x_{i+1} - x_i$ are not smaller than a certain value.

The iterative process is constructed in the following way. First the Galerkin method is applied to the original equation (1) and the system of the algebraic equations (5) yields the first iteration for $y_i (i = 1, ..., n)$. Then the values $y_i(i, ..., n)$ are splined and the first iteration of Δy, $\frac{d(\Delta y)}{dx}$ and $\frac{d^2(\Delta y)}{dx^2}$ can be obtained for every $x \in (0, 1)$

$$\Delta y = y_s(x) - \tilde{y}(x), \tag{22}$$

$$\frac{d(\Delta y)}{dx} = \frac{dy_s(x)}{dx} - \frac{d\tilde{y}(x)}{dx}, \tag{23}$$

$$\frac{d^2(\Delta y)}{dx^2} = \frac{d^2 y_s(x)}{dx^2} - \frac{d^2\tilde{y}(x)}{dx^2} \tag{24}$$

where $y_s(x)$, $\frac{dy_s(x)}{dx}$ are obtained using the splining technique and $\frac{d^2 y_s(x)}{dx^2}$ is computed from equation (1) using these values. The need to compute $\frac{d^2 y_s(x)}{dx^2}$ directly from (1) arises due to the fact that in case when $y_s(x)$ is a steep function the splining technique does not yield acceptable accuracy for the second derivative. Evidently, for parabolic shape functions the formulas (23) and (24) do not make sense at the points $x = x_i$ $(i = 3, 5, ..., n - 2)$.

The modified equation for the second iteration is:

$$f\left\{\left[\frac{d^2 y}{dx^2} + \frac{d^2(\Delta y)}{dx^2}\right], \ \left[\frac{dy}{dx} + \frac{d(\Delta y)}{dx}\right], [y + \Delta y], x\right\} = 0 \tag{25}$$

or

$$f_m\left(\frac{d^2 y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0.$$

If the equation (1) were linear and homogeneous, then the following would be true:

$$f_m\left(\frac{d^2 y}{dx^2}, \frac{dy}{dx}, y, x\right) = f\left(\frac{d^2 y}{dx^2}, \frac{dy}{dx}, y, x\right) + f\left[\frac{d^2(\Delta y)}{dx^2}, \frac{d(\Delta y)}{dy}, \Delta y, x\right]. \tag{26}$$

Now applying the Galerkin method to the modified equation (20) we can find the new values $y_i (i = 1, 2, .., n)$ and new functions Δy, $\frac{d(\Delta y)}{dx}$ and $\frac{d^2(\Delta y)}{dx^2}$ etc. When the process converges (if it converges) we will find the highly accurate values for $y_i (i = 1, ..., n)$. The error is caused only by inaccuracies of the splining technique. The inaccuracies of the splining technique with regard to $y_s(x)$ and $\frac{dy_s(x)}{dx}$ can be made smaller than any desired value, and the value of $\frac{d^2 y_s(x)}{dx^2}$ can be computed directly from (1) with the same accuracy, Therefore, the inaccuracy of the Galerkin-Gokhman method can be made smaller than any desired value.

The following question arises naturally in this connection: why not use the splines as the shape functions taking for coefficients a_i in (6) the unknown values of y_i and $\left.\frac{dy}{dx}\right|_i$. The answer is simple. First, the second derivatives of the solution in a form of the spline with predefined values of $\left.\frac{dy}{dx}\right|_i$ are discontinuous at the nodal points. Second, when the solution $y = y(x)$ of the equation (1) is a steep function, the cubic spline, this time with continuous second derivatives at the nodal points, fails to represent the second derivatives in the zone of large first, second and third derivatives of the solution (see section 3). Therefore, the splining technique cannot be used directly for finite element analysis in applications to second order differential equations whose solutions are steep functions. Direct application of fifth order splines was not analyzed, because of the large number of unknowns. In two dimensions we have a 5 fold increase in the number of unknowns going from cubic to quintic splines.

2. GALERKIN APPROXIMATION WITH CONTINUOUS FIRST AND SECOND DERIVATIVES

In order for the function $\tilde{y} = \tilde{y}(x)$ to pass through the two nodal point elements with continuous the first and the second derivatives one can use the cubic spline ($i = 1, 2, ..., n-1$, where n is the number of nodal points):

$$y^{(i)} = y_i G_i + y_{i+1} G_{i+1} \qquad x_i \leq x \leq x_{i+1} \tag{27}$$

where

$$G_i = 1 + a_{1,i}\lambda + a_{2,i}\lambda^2 - (1 + a_{1,i} + a_{2,i})\lambda^3$$
$$G_{i+1} = a_{1,i}\lambda + a_{2,i}\lambda^2 + (1 - a_{1,i} - a_{2,i})\lambda^3$$

and

$$\lambda = \frac{x - x_i}{\Delta x_i}.$$

It is easy to see from (27) that

$$y^{(i)} \mid_{\lambda=0} = y_i \quad (x = x_i)$$
$$y^{(i)} \mid_{\lambda=1} = y_{i+1} \quad (x = x_{i+1})$$

Differentiating (27) we can find

$$\frac{dy^{(i)}}{dx} = \frac{1}{\Delta x_i}\left(y_i \frac{dG_i}{d\lambda} + y_{i+1}\frac{dG_{i+1}}{d\lambda}\right) \tag{28}$$

where

$$\frac{dG_i}{d\lambda} = a_{1,i} + 2a_{2,i}\lambda - 3(1 + a_{1,i} + a_{2,i})\lambda^2,$$
$$\frac{dG_{i+1}}{d\lambda} = a_{1,i} + 2a_{2,i}\lambda + 3(1 - a_{1,i} - a_{2,i})\lambda^2.$$

Now differentiating (28) we find that

$$\frac{d^2y^{(i)}}{dx^2} = \frac{1}{(\Delta x_i)^2}\left(y_i \frac{d^2G_i}{d\lambda^2} + y_{i+1}\frac{d^2G_{i+1}}{d\lambda^2}\right) \tag{29}$$

where

$$\frac{d^2G_i}{d\lambda^2} = 2a_{2,i} - 6(1 + a_{1,i} + a_{2,i})\lambda,$$

$$\frac{d^2G_{i+1}}{d\lambda^2} = 2a_{2,i} + 6(1 - a_{1,i} - a_{2,i})\lambda.$$

The $2(n-1)$ unknown values of $a_{1,i}$ and $a_{2,i}$ $(i = 1, 2, ..., n-1)$ can be found from the $2(n-2)$ continuity conditions for the first and second derivatives at nodal points (excluding the first and last ones):

$$\left.\frac{dy^{(i)}}{dx}\right|_{\lambda=1} = \left.\frac{dy^{(i+1)}}{dx}\right|_{\lambda=0},$$
$$\left.\frac{d^2y^{(i)}}{dx^2}\right|_{\lambda=1} = \left.\frac{d^2y^{(i+1)}}{dx^2}\right|_{\lambda=0}. \tag{30}$$

and two additional conditions at the first and last nodal points (specification of the first derivatives $\left.\frac{dy}{dx}\right|_1$ and $\left.\frac{dy}{dx}\right|_n$.

The final form of the system of the linear algebraic equations for the determination of $a_{1,i}$ and $a_{2,i}$ $(i = 1, ..., n-1)$ follows:

$$a_{1,1}(y_1 + y_2) = \left.\frac{dy}{dx}\right|_1 \Delta x_1,$$

$$a_{1,n-1}[2(y_{n-1} + y_n)] + a_{2,n-1}(y_{n-1} + y_n) = -\left.\frac{dy}{dx}\right|_1 \Delta x_{n-1} + 3(y_n - y_{n-1})$$

and for $i = 2, 3, ..., n-1$ (31)

$$a_{1,i-1}[2(y_{i-1} + y_i)] + a_{2,i-1}(y_{i-1} + y_i) + a_{1,i}[c_i(y_i + y_{i+1})] = 3(y_i - y_{i-1}),$$
$$a_{1,i-1}[3(y_{i-1} + y_i)] + a_{2,i-1}[2(y_{i-1} + y_i)] + a_{2,i}[c_i^2(y_i + y_{i+1})] = 3(y_i - y_{i-1})$$

where

$$c_i = \frac{\Delta x_{i-1}}{\Delta x_i}.$$

The shape functions yielding the Galerkin approximation with continuous first and second derivatives change during the iterative process since the values of $a_{1,i}$ and $a_{2,i}$ depend on the distribution of y_i. In this situation the first iteration for $a_{1,i}$ and $a_{2,i}$ must be obtained using the solution obtained with conventional shape functions, linear or parabolic.

A differential equation, whose solution is not a steep function can be solved using shape functions described in this section without the modification employed in the Galerkin-Gokhman method. However, as was mentioned above we will show in section 3, that for a second order ODE whose solution is a steep function this approach will not work, because the values of the second derivative for the solution cannot be obtained with acceptable accuracy by the splining technique.

3. COMPARISON OF THE GALERKIN AND THE GALERKIN-GOKHMAN METHODS IN APPLICATION TO TWO ORDINARY DIFFERENTIAL EQUATIONS

The first application used to compare the Galerkin and the Galerkin-Gokhman methods was an equation often employed in numerical analysis for comparison of different numerical methods [6]:

$$y - \frac{dy}{dx} = 0, \ \ y(0) = 1, [0 \leq x \leq 1]. \tag{32}$$

The modified equation is:

$$y - \frac{dy}{dx} + \Delta y - \frac{d\Delta y}{dx} = 0 \tag{33}$$

where:

$$\Delta y = y_s(x) - \tilde{y}(x).$$

The segment $[0, 1]$ was divided into four equal elements with five nodal points $x = 0.00, 0.25, 0.50, 0.75, 1.00$. The finite element solution was based on linear shape functions. The functions $\tilde{y} = \tilde{y}(x)$ and $y_s = y_s(x)$ are shown in Figure 1. The results of computation are presented in Table 1. The Galerkin method yields a solution with the maximum relative error of 0.03288 at $x = 0.25$ and the maximum error of the Galerkin-Gokhman solution for the fourth iteration is 0.00304. For the thirty second iteration the maximum error is 0.00003 at the same point $x = 0.25$. In addition, it can be seen from Table 1 that the maximum error in computation of the derivative for the fourth iteration is 0.00844. In the case of solution of the equation (32) the splining technique could be applied directly (Galerkin approximation with unknown values for y_i and $\left.\frac{dy}{dx}\right|_i$) because this is the first-order equation with gradual change of solution along the entire domain.

The second application used to compare the Galerkin and the Galerkin- Gokhman methods was a second order nonlinear ordinary differential equation describing the fully developed turbulent flow of incompressible fluid in a pipe [1]:

$$\frac{d}{dr}\left(r\nu_e\frac{dv_z}{dr}\right) - \frac{r}{\rho}\frac{dp}{dz} = 0 \tag{34}$$

where

$$\nu_e = \nu - l^2\frac{dv_z}{dr}$$

and

$\frac{dp}{dz}$ is the pressure gradient along the pipe,

r is the radius of the point,

v_z is velocity of the flow,

ρ is the density of the fluid,

ν is the Newtonian viscosity,

l is the Prandtl mixing length.

The boundary conditions for the solution of (34) are:

$$
\begin{aligned}
&v_z(R) = 0, \\
&\left.\frac{dv_z}{dr}\right|_{r=0} = 0
\end{aligned}
\tag{35}
$$

where R is the radius of the pipe.

The mixing length in (34) was taken according to Nikuradse with the Van Driest correction [4]:

$$
\frac{l}{R} = \left[0.14 - 0.08\left(\frac{r}{R}\right)^2 - 0.06\left(\frac{r}{R}\right)^4\right]\left\{1 - exp\left[\frac{v_*(R-r)}{26\nu}\right]\right\} \tag{36}
$$

where $v_* = (\tau_0/\rho)^{0.5}$ (τ_0 is the tangential stress at the wall).

The modified equation is:

$$
\frac{d}{dr}\left\{r\left[\nu_e - l^2\frac{d(\Delta v_z)}{dr}\right]\left[\frac{dv_z}{dr} + \frac{d(\Delta v_z)}{dr}\right]\right\} - \frac{r}{\rho}\frac{dp}{dz} = 0 \tag{37}
$$

where $\Delta v_z = v_{zs} - \tilde{v}_z$.

The Galerkin integrals (projections to the basis functions G_i) of the equation (37) are ($i = 1, ..., n-1$):

$$
\begin{aligned}
&\int_0^R \frac{d}{dr}\left\{r\left[\tilde{\nu}_e - l^2\frac{d(\Delta v_z)}{dr}\right]\left[\frac{d\tilde{v}_z}{dr} + \frac{d(\Delta v_z)}{dr}\right]\right\} G_i(r)dr - \\
&\quad \int_0^R \frac{r}{\rho}\frac{dp}{dz}G_i(r)dr = 0
\end{aligned}
\tag{38}
$$

where:

$$
\begin{aligned}
&\frac{d\tilde{v}_z}{dr} = \sum_{j=1}^{n} v_{zj}\frac{dG_j(r)}{dr}, \\
&\tilde{\nu}_e = \nu - l^2\frac{d\tilde{v}_z}{dr}.
\end{aligned}
$$

Integrating by parts the first integral in (38) we obtain the final system of second order algebraic equations for determination of v_{zi} ($i = 1, ..., n-1$):

$$
\int_0^R \left[A\left(\frac{d\tilde{v}_z}{dr}\right)^2 + B\frac{d\tilde{v}_z}{dr} + C\right] dr = 0 \tag{39}
$$

where:

$$
\begin{aligned}
&A = -l^2\frac{dG_j(r)}{dr}r, \\
&B = \left[\nu - 2l^2\frac{d(\Delta v_z)}{dr}\right]\frac{dG_j(r)}{dr}r,
\end{aligned}
$$

$$C = \frac{r}{\rho}\frac{dp}{dz}G_i(r) + \left[\nu - l^2\frac{d(\Delta v_z)}{dr}\right]\frac{d(\Delta v_z)}{dr}\frac{dG_j(r)}{dr}r.$$

The system of equations (39) yields a solution for the equation (34) using the Galerkin-Gokhman method. The first iteration in obtaining this solution (for $\frac{d(\Delta v_z)}{dr} = 0$) is the solution according to the Galerkin method.

The global shape functions in (39) were taken as parabolic (see formulas (10)–(14)). The Galerkin and the Galerkin-Gokhman were compared with a highly accurate solution of (34) obtained exactly the same way as in [4].

The comparison was performed for Reynolds number Re= 10^7 and for the relative length of element at wall $\Delta\eta = \frac{R - r_{n-2}}{R}$ in the range 0.00005–0.00060. The results are presented in Figure 2. Figure 2 shows $(\delta v_z)_{max}$, the maximum relative error in computation of v_z, versus $\Delta\eta$ using the Galerkin and the Galerkin-Gokhman methods. The Galerkin method error grows drastically with $\Delta\eta$ (at $\Delta\eta = 0.00010$, $(\delta v_z)_{max} = 0.02123$; at $\Delta\eta = 0.00020$, $(\delta v_z)_{max} = 0.18572$). On the other hand the Galerkin-Gokhman method error grows very gradually with $\Delta\eta$ (at $\Delta\eta = 0.00010$, $(\delta v_z)_{max} = 0.00681$; at $\Delta\eta = 0.00020$, $(\delta v_z)_{max} = 0.00814$). The minimum value of $(\delta v_z)_{max}$ for the Galerkin method is 0.00807 and for the Galerkin-Gokhman method 0.00138.

The values of $\frac{dv_{zs}}{dr}$ for the computation of $\frac{d(\Delta v_z)}{dr}$ occurring in the solution of the algebraic system (39) for every iteration (except the first, where $\frac{d(\Delta v_z)}{dr} = 0$) were obtained using a parametric spline:

$$\begin{aligned} &\eta = \eta(s) \\ &v_z = v_z(s) \end{aligned} \tag{40}$$

where:

$\eta = \frac{r}{R}$ is relative radius
s is the length along the spline

The formulas relating $\frac{dv_z}{ds}$ and $\frac{d\eta}{ds}$ to $\frac{dv_z}{dr}$ are:

$$\begin{aligned} &\frac{dv_z}{ds} = \frac{\frac{dv_z}{d\eta}}{\sqrt{1 + \left(\frac{dv_z}{d\eta}\right)^2}}. \\ &\frac{d\eta}{ds} = \frac{1}{\sqrt{1 + \left(\frac{dv_z}{d\eta}\right)^2}} \end{aligned} \tag{41}$$

where:

$$\frac{dv_z}{d\eta} = R\frac{dv_z}{dr}.$$

The spline (40) passes through the points $[\eta_i,\, v_z(\eta_i)]$. The values of $v_z(\eta_i)$ were computed in the previous iteration. The values of $\frac{dv_z}{dr}$ at $\eta = 0$ and $\eta = 1$ (boundary conditions for spline computation) were determined as follows. At $\eta = 0\,(r = 0)$, $\frac{dv_z}{dr} = 0$. This is one of the boundary conditions for (34).

At $\eta = 1\,(r = R)$, using the momentum equation [4]:

$$\left.\frac{dv_z}{dr}\right|_{r=R} = \frac{R}{2\rho\nu}\frac{dp}{dz}. \tag{42}$$

Table 2 compares the accuracy of computing $\frac{dv_z}{d\eta}$ using the parametric spline (40) and spline $v_z = v_z(\eta)$. Both splines were obtained using the values of $v_z(\eta_i)$ given by highly accurate solution [4]. This table shows the results only for the nodal points η_i close to the pipe wall in the interval [0.998800, 1.000000], where the highest inaccuracy occurs. This comparison shows the superiority of the parametric spline (40).

It is easy to see that for the system of equations (39) the second derivative $\frac{d^2(\Delta v_z)}{dr^2}$ does not appear in computations and, therefore, the accuracy of computing $\frac{d^2 v_{zs}}{dr^2}$ is irrelevant in the case of fully developed flow. However, in the case of developing flow in a pipe, considered in the next section, the second derivatives do occur and the accuracy of their computation using the splining technique is very important for the overall accuracy of the solution. As was mentioned in section 1 the splining technique by itself cannot yield the values of the second derivative $\frac{d^2 v_z}{dr^2}$ with acceptable accuracy and, therefore, the second derivative must be computed directly from (34) using the value of the first derivative $\frac{dv_z}{dr}$ given by the parametric spline (40).

Table 3 shows the comparison of accuracy for $\frac{d^2 v_z}{d\eta^2}$ computed using the parametric spline (40) and computed directly from equation (34) using the values of $\frac{dv_z}{d\eta}$ given by the spline (40). The spline (40) was obtained using the values of $v_z(\eta_i)$ given by the highly accurate solution [4].

The comparison in Table 3 is shown only for the nodal points η_i in the interval [0.998800, 1.000000] where the highest inaccuracy occurs. Table 3 shows that the spline of the third order is not capable of representing the values of velocity at the points which are near the pipe wall. At the same, time Table 3 shows that the values of the second derivative obtained directly from equation (34) using the values of the first derivative given by the spline (40) have acceptable accuracy.

The exact values of $\frac{dv_z}{d\eta}$ and $\frac{d^2 v_z}{d\eta^2}$ in Tables 2 and 3 were computed using the formulas in [4], where the highly accurate solution of the equation (34) is described.

4. FORMULATION OF THE GALERKIN-GOKHMAN METHOD FOR THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

In this section we demonstrate the application of the Galerkin-Gokhman method to a system of partial differential equations using the example of Navier-Stokes equations for developing flow in a pipe.

The Navier-Stokes equations [7] for steady turbulent flow of incompressible fluid (with the z axis being the axis of symmetry) in the absence of mass forces are:

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu_e \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\,\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) +$$
$$2\frac{\partial \nu_e}{\partial r}\frac{\partial v_r}{\partial r} + \frac{\partial \nu_e}{\partial z}\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right), \tag{43}$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\,\frac{\partial p}{\partial z} + \nu_e \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\,\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) +$$
$$2\frac{\partial \nu_e}{\partial z}\,\frac{\partial v_z}{\partial z} + \frac{\partial \nu_e}{\partial r}\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \tag{44}$$

where, according to Boussinesq, the effective kinematic viscosity:

$$\nu_e = \nu + \nu_t \tag{45}$$

and

ν is Newtonian viscosity,

ν_t is turbulent viscosity.

The continuity equation is:

$$\frac{1}{r}\,\frac{\partial (v_r r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0. \tag{46}$$

Turbulent viscosity in (45):

$$\nu_t = f\left(l, \frac{\partial v_z}{\partial r}, \frac{\partial v_r}{\partial z} \right) \tag{47}$$

where:

$l = l(r, z)$ is the Prandtl mixing length.

Therefore:

$$\frac{\partial \nu_e}{\partial r} = g\left(l, \frac{\partial l}{\partial r}, \frac{\partial v_z}{\partial r}, \frac{\partial^2 v_z}{\partial r^2}, \frac{\partial v_r}{\partial z}, \frac{\partial^2 v_r}{\partial r\,\partial z} \right), \tag{48}$$
$$\frac{\partial \nu_e}{\partial z} = h\left(l, \frac{\partial l}{\partial z}, \frac{\partial v_z}{\partial r}, \frac{\partial^2 v_z}{\partial z\,\partial r}, \frac{\partial v_r}{\partial z}, \frac{\partial^2 v_r}{\partial z^2} \right). \tag{49}$$

For each iteration, excluding the first, let (see section 1):

$$\Delta v_z = v_{zs} - \tilde{v}_z$$
$$\Delta v_r = v_{rs} - \tilde{v}_r \tag{50}$$

$$\Delta p = p_s - \tilde{p}$$

where:

$\tilde{v}_z$, $\tilde{v}_r$ and $\tilde{p}$ are the Galerkin approximations

v_{zs}, v_{rs} and p_s are the spline representations

In a case of the Galerkin approximation with discontinuous first and second derivatives the well known conventional formulae can be applied for $\tilde{v}_z\,, \tilde{v}_r$ (eight nodal points element) and $\tilde{p}$ (four nodal points element) see [7].

The functions for v_{zs}, v_{rs} and p_s are defined using the formulae for splining technique presentation for the function of two variables [8]. All approximations $\tilde{v}_z$, $\tilde{v}_r$ and $\tilde{p}$ and v_{zs}, v_{rs} and p_s are based on the values of v_z, v_r and p determined in the previous iteration. For the first iteration Δv_z, Δv_r and Δp are taken to be zero. Therefore the first iteration is the solution of the unmodified equations (43), (44) and (45).

As follows from the section 1, the modified Navier-Stokes equations are:

$$\begin{aligned}
&(v_r + \Delta v_r)\left[\frac{\partial v_r}{\partial r} + \frac{\partial(\Delta v_r)}{\partial r}\right] + (v_z + \Delta v_z)\left[\frac{\partial v_r}{\partial z} + \frac{\partial(\Delta v_r)}{\partial z}\right] = \\
&\quad -\frac{1}{\rho}\left[\frac{\partial p}{\partial r} + \frac{\partial(\Delta p)}{\partial r}\right] + \nu_{em}\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} + f_r\right) + \\
&\quad 2\,\frac{\partial \nu_{em}}{\partial r}\left[\frac{\partial v_r}{\partial r} + \frac{\partial(\Delta v_r)}{\partial r}\right] + \frac{\partial \nu_{em}}{\partial z}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} + \frac{\partial(\Delta v_z)}{\partial r} + \frac{\partial(\Delta v_r)}{\partial z}\right],
\end{aligned} \tag{51}$$

$$\begin{aligned}
&(v_r + \Delta v_r)\left[\frac{\partial v_z}{\partial r} + \frac{\partial(\Delta v_z)}{\partial r}\right] + (v_z + \Delta v_z)\left[\frac{\partial v_z}{\partial z} + \frac{\partial(\Delta v_z)}{\partial z}\right] = \\
&\quad -\frac{1}{\rho}\left[\frac{\partial p}{\partial z} + \frac{\partial(\Delta p)}{\partial z}\right] + \nu_{em}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} + f_z\right) + \\
&\quad 2\,\frac{\partial \nu_{em}}{\partial z}\left[\frac{\partial v_z}{\partial z} + \frac{\partial(\Delta v_z)}{\partial z}\right] + \frac{\partial \nu_{em}}{\partial r}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} + \frac{\partial(\Delta v_z)}{\partial r} + \frac{\partial(\Delta v_r)}{\partial z}\right]
\end{aligned} \tag{52}$$

where:

$$\nu_{em} = \nu + f\left[l, \frac{\partial v_z}{\partial r} + \frac{\partial(\Delta v_z)}{\partial r}, \frac{\partial v_r}{\partial z} + \frac{\partial(\Delta v_r)}{\partial z}\right], \tag{53}$$

$$\begin{aligned}
\frac{\partial \nu_e}{\partial r} = g\Bigg[&l, \frac{\partial l}{\partial r}, \frac{\partial v_z}{\partial r} + \frac{\partial(\Delta v_z)}{\partial r},\ \frac{\partial^2 v_z}{\partial r^2} + \frac{\partial^2(\Delta v_z)}{\partial r^2}, \\
&\frac{\partial v_r}{\partial z} + \frac{\partial(\Delta v_r)}{\partial z}, \frac{\partial^2 v_r}{\partial r\,\partial z} + \frac{\partial^2(\Delta v_r)}{\partial r\,\partial z}\Bigg],
\end{aligned} \tag{54}$$

$$\begin{aligned}
\frac{\partial \nu_e}{\partial z} = h\Bigg[&l, \frac{\partial l}{\partial z}, \frac{\partial v_z}{\partial r} + \frac{\partial(\Delta v_z)}{\partial r},\ \frac{\partial^2 v_z}{\partial z\,\partial r} + \frac{\partial^2(\Delta v_z)}{\partial z\,\partial r}, \\
&\frac{\partial v_r}{\partial z} + \frac{\partial(\Delta v_r)}{\partial z}, \frac{\partial^2 v_r}{\partial z^2} + \frac{\partial^2(\Delta v_r)}{\partial z^2}\Bigg],
\end{aligned} \tag{55}$$

$$f_r = \frac{\partial^2(\Delta v_r)}{\partial r^2} + \frac{1}{r}\frac{\partial(\Delta v_r)}{\partial r} - \frac{(\Delta v_r)}{r^2} + \frac{\partial^2(\Delta v_r)}{\partial z^2}, \tag{56}$$

$$f_z = \frac{\partial^2(\Delta v_z)}{\partial r^2} + \frac{1}{r}\frac{\partial(\Delta v_z)}{\partial r} + \frac{\partial^2(\Delta v_z)}{\partial z^2}. \tag{57}$$

The modified continuity equation is:

$$\frac{1}{r}\left[\frac{\partial(v_r r)}{\partial r} + \frac{\partial(\Delta v_r r)}{\partial r}\right] + \frac{\partial v_z}{\partial z} + \frac{\partial(\Delta v_z)}{\partial z} = 0. \tag{58}$$

The domain for the solution of Navier-Stokes equations describing developing flow in a pipe is shown on Figure 3. Here z_{FD} is the value of z where the flow is considered to be fully developed. The value z_{FD} is usually taken to be up to 150 pipe diameters [7]. In order to apply the splining technique to the solution obtained as a result of the current iteration, we need to know the appropriate values for the partial derivatives along the boundaries of the domain. These boundary conditions are evaluated below.

1. Inlet to the pipe ($0 \le r \le R,\ z = 0$)

Here we need to evaluate $\left.\frac{\partial v_z}{\partial z}\right|_I$, $\left.\frac{\partial v_r}{\partial z}\right|_I$ and $\left.\frac{\partial p}{\partial z}\right|_I$.

We assume that at the inlet $v_{rI} = 0$, $v_{zI} =$ const and $p_I =$ const. Therefore, we have:

$$\begin{aligned} &\left.\frac{\partial v_r}{\partial r}\right|_I = 0,\ \left.\frac{\partial (v_r r)}{\partial r}\right|_I = 0,\ \left.\frac{\partial^2 v_r}{\partial r^2}\right|_I = 0, \\ &\left.\frac{\partial v_z}{\partial r}\right|_I = 0,\ \left.\frac{\partial^2 v_z}{\partial r^2}\right|_I = 0. \end{aligned} \tag{59}$$

Now, using (59) we obtain from (46) that:

$$\left.\frac{\partial v_z}{\partial z}\right|_I = 0. \tag{60}$$

In order to simplify the evaluation we will assume that:

$$\left.\frac{\partial v_r}{\partial z}\right|_I = 0. \tag{61}$$

It follows from (46), after differentiation with respect to z, and from (61) that:

$$\left.\frac{\partial^2 v_z}{\partial z^2}\right|_I = 0 \tag{62}$$

and finally from equation (44) using (59), (60), (61), and(62):

$$\left.\frac{\partial p}{\partial z}\right|_I = 0.$$

The assumption (61) is not required. However, its absence would lead to a cumbersome analysis of the iterative process for the evaluation of boundary conditions, which falls outside the scope of present paper.

2. Outlet of the pipe ($0 \leq r \leq R,\ z = z_{FD}$)

Here we also need to evaluate $\left.\frac{\partial v_z}{\partial z}\right|_O$, $\left.\frac{\partial v_r}{\partial z}\right|_O$ and $\left.\frac{\partial p}{\partial z}\right|_O$.
Since the flow at the outlet is fully developed, we have:

$$\left.\frac{\partial v_r}{\partial z}\right|_O = 0,\ \left.\frac{\partial v_r}{\partial z}\right|_O = 0,\ \left.\frac{\partial p}{\partial z}\right|_O = \text{const.}$$

However, the value of $\left.\frac{\partial p}{\partial z}\right|_O$ is not known, so the equivalent condition $\left.\frac{\partial^2 p}{\partial z^2}\right|_O = 0$ is used.

3. Axis of the pipe ($r = 0,\ 0 \leq z \leq z_{FD}$)

The obvious boundary conditions are:

$$\left.\frac{\partial v_z}{\partial r}\right|_A = 0,\ \left.\frac{\partial v_r}{\partial r}\right|_A = 0,\ \left.\frac{\partial p}{\partial r}\right|_A = 0.$$

4. The wall of the pipe ($r = R,\ 0 \leq z \leq z_{FD}$)

We need to evaluate the values of $\left.\frac{\partial v_z}{\partial r}\right|_W$, $\left.\frac{\partial v_r}{\partial r}\right|_W$ and $\left.\frac{\partial p}{\partial r}\right|_W$.
Along the wall we have:

$$\begin{aligned} &v_{zW} = 0,\ \left.\frac{\partial v_z}{\partial z}\right|_W = 0,\ \left.\frac{\partial^2 v_z}{\partial z^2}\right|_W = 0, \\ &v_{rW} = 0,\ \left.\frac{\partial v_r}{\partial z}\right|_W = 0,\ \left.\frac{\partial^2 v_r}{\partial z^2}\right|_W = 0 \end{aligned} \tag{63}$$

and because of the laminar sublayer:

$$\nu_{eW} = \nu,\ \left.\frac{\partial \nu_e}{\partial z}\right|_W = 0. \tag{64}$$

On the other, hand from (46) and (63) it follows that:

$$\left.\frac{\partial v_r}{\partial r}\right|_W = 0 \tag{65}$$

Therefore, using (63), (64), and (65) equation (43) is reduced to:

$$\frac{1}{\rho}\left.\frac{\partial p}{\partial r}\right|_W = \nu \left.\frac{\partial^2 v_r}{\partial r^2}\right|_W. \tag{66}$$

The value of $\left.\frac{\partial^2 v_r}{\partial r^2}\right|_W$ in (66) can be obtained after splining v_r using the boundary condition (65).

The last boundary value to be obtained is $\left.\frac{\partial v_z}{\partial r}\right|_W$. For this purpose we use the momentum equation along z (see Figure 3.):

$$\tau = \frac{1}{R} \int_0^R \frac{\partial p}{\partial z} r dr. \tag{67}$$

However, at the pipe wall:

$$\tau = \rho\nu \left. \frac{\partial v_z}{\partial r} \right|_W . \tag{68}$$

Therefore, using (67) and (68) we get:

$$\left. \frac{\partial v_z}{\partial r} \right|_W = \frac{1}{\rho\nu R} \int_0^R \frac{\partial p}{\partial z} r dr. \tag{69}$$

The function $\frac{\partial p}{\partial r} = \frac{\partial p}{\partial r}(r)$ in (69) is the result of the previous iteration.

The functions v_{zs}, v_{rs}, p_s and their first derivatives with respect to r and z can be obtained with acceptable accuracy using the splining presentations of v_z, v_r and p by themselves (see section 3). Since v_r is small and both v_z and v_r do not change drastically in the z-direction, the following second derivatives of velocity components can also be computed with acceptable accuracy using the splining presentations by themselves:

$$\frac{\partial^2 (v_r)_s}{\partial z^2}, \frac{\partial^2 (v_z)_s}{\partial z^2}, \frac{\partial^2 (v_r)_s}{\partial r \, \partial z}.$$

On the other hand the derivatives:

$$\frac{\partial^2 (v_z)_s}{\partial r^2}, \frac{\partial^2 (v_r)_s}{\partial r^2}, \frac{\partial^2 (v_z)_s}{\partial r \, \partial z}$$

must be computed using the values of all other derivatives obtained using the splining technique by itself from Navier-Stokes equations (43), (44) and from the equation obtained by differentiating the continuity equation (46) with respect to r:

$$\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_z}{\partial r \, \partial z} = 0.$$

As shown in section 3, this approach leads to acceptable accuracy in computation.

CONCLUSION

The new method for solving ordinary and partial differential equations using the finite element technique when the solution is a steep function (the Galerkin-Gokhman method) provides significantly better accuracy for the same discretization than the conventional Galerkin procedure for the second order nonlinear ordinary differential equation describing fully developed flow in a pipe.

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TABLE 1

Relative errors $\delta(y)$ and $\delta\left(\frac{dy}{dx}\right)$ in computation of y and $\frac{dy}{dx}$ for the equation: $y - \frac{dy}{dx} = 0$ using the Galerkin-Gokhman method.

Iteration 1 (The Galerkin Method)

x	y	$\delta(y)$

0.00000	1.00000	0.00000
0.25000	1.24181	0.03288
0.50000	1.63339	0.00930
0.75000	2.06156	0.02619
1.00000	2.68002	0.01407

Iteration 4

x	y	$\delta(y)$	$\frac{dy}{dx}$	$\delta\left(\frac{dy}{dx}\right)$
0.00000	1.00000	0.00000	1.00000	0.00000
0.25000	1.28012	0.00304	1.29486	0.00844
0.50000	1.64622	0.00152	1.64473	0.00242
0.75000	2.11277	0.00200	2.12747	0.00495
1.00000	2.71293	0.00197	2.72534	0.00260

Iteration 32

x	y	$\delta(y)$	$\frac{dy}{dx}$	$\delta\left(\frac{dy}{dx}\right)$
0.00000	1.00000	0.00000	1.00000	0.00000
0.25000	1.28407	0.00003	1.28404	0.00001
0.50000	1.64874	0.00001	1.64871	0.00001
0.75000	2.11706	0.00003	2.11701	0.00001
1.00000	2.71832	0.00001	2.71833	0.00002

TABLE 2

Relative error $\delta\left(\frac{dv_z}{d\eta}\right)$ in splining representation of $\frac{dv_z}{d\eta}$ for Re=10^7.

η_i Nodal Points	$\frac{dv_z}{d\eta}$ Exact Values	$\delta\left(\frac{dv_z}{d\eta}\right)$ Spline: $v_z = v_z(\eta)$	$\delta\left(\frac{dv_z}{d\eta}\right)$ Parametric Spline
0.998800	-57.701622	0.00539	0.00015
0.999100	-77.014101	0.01461	0.00031
0.999400	-117.014101	0.02507	0.00091
0.999600	-186.464550	0.05778	0.00303
0.999800	-466.337151	0.01844	0.00973
0.999850	-708.084207	0.01222	0.00737
0.999900	-1281.849549	0.00502	0.00653
0.999925	-1903.870475	0.00493	0.00377
0.999950	-3000.130832	0.01219	0.00218
0.999975	-4204.519965	0.00766	0.00515
1.000000	-4430.460863	0.00000	0.00000

TABLE 3

Relative error $\delta\left(\frac{d^2v_z}{d\eta^2}\right)$ in computation of $\frac{dv_z}{d\eta}$ using parametric spline for Re=10^7.

η_i Nodal Points	$\frac{d^2v_z}{d\eta^2}$ Exact Values	$\delta\left(\frac{d^2v_z}{d\eta^2}\right)$ Spline Only	$\delta\left(\frac{d^2v_z}{d\eta^2}\right)$ Spline and Equation
0.998800	-47961.956	0.00633	0.00015
0.999100	-86676.235	0.01181	0.00031
0.999400	-211604.762	0.00500	0.00092
0.999600	-561704.632	0.03962	0.00309
0.999800	-3324546.134	0.06604	0.01027
0.999850	-6940590.540	0.03547	0.00801
0.999900	-18371229.352	0.06016	0.00763
0.999925	-32965399.067	0.03698	0.00480
0.999950	-54317924.688	0.08902	0.00330
0.999975	-28569779.891	0.27617	0.00986
1.000000	-4430.461	1472.45067	0.00000

FIGURE CAPTIONS

Figure 1. The first iteration for the solution of equation $y - \frac{dy}{dx} = 0$ using the Galerkin-Gokhman method.

Figure 2. Parametric study of the accuracy of the solution for the equation describing fully developed flow in a pipe using the Galerkin and Galerkin-Gokhman methods at Re=10^7.

Figure 3. Developing flow in a pipe.