DETERMINATION OF STREAMLINES ON STREAM SURFACE WITH DISCRETELY DEFINED GEOMETRY AND VELOCITY DISTRIBUTION

Alexander Gokhman
John Leitner

Research and Development Center
Allis-Chalmers Fluid Products Company
York, Pennsylvania

ABSTRACT

Numerical generation of streamlines along an arbitrary stream surface is discussed in the present work. Using a given set of nodal Cartesian coordinates, which define a stream surface, and the associated velocities at each node, a new non-orthogonal, curvilinear coordinate system (p,q) is devised along the surface. On the basis of this curvilinear system a piecewise, analytically defined surface is generated passing through the p and q coordinate lines. Using the velocity components at each nodal point, a piecewise, analytical representation of the velocity distribution (as a function of p and q), along the surface is determined. Using the differential geometry of the surface and velocity distribution along it, a set of streamlines is calculated along with an additional set of streamline orthogonal lines.

NOMENCLATURE

a	Radius of cylinder
a, b, c	Coefficients in the canonical equation of ellipsoid
$\hat{e}$	Unit vector for curvilinear coordinate system
G_{01}, G_{11}	Coefficients at variables in Hermite Interpolant Polynomial for j splines
G_{02}, G_{12}	Coefficients at derivatives in Hermite Interpolant Polynomial for j splines
H_{01}, H_{11}	Coefficients at variables in Hermite Interpolant Polynomial for i splines
H_{02}, H_{12}	Coefficients at derivatives in Hermite Interpolant Polynomial for i splines
H	Lamme coefficient
$\hat{i}$, $\hat{j}$, $\hat{k}$	Unit vectors for the Cartesian coordinate system
ℓ_s	Length along streamline on the stream surface
ℓ and s	Length along i and j splines respectively
M	Dipol moment intensity
p and q	Non-orthogonal curvilinear coordinate system on the stream surface
p_o and q_o	Orthogonal curvilinear coordinate system on the stream surface
R	Radius of sphere
U	Velocity of the uniform flow in the infinity
$\vec{V}$	Velocity vector of the flow along the stream surface
W	Complex potential
x, y, z	Cartesian coordinates of the point on the stream surface
α	Viewing angle of the element on the stream surface
$\Delta_i(F)$	Increment of value F along the segment i
$\delta(F)$	Error in computation of value F
λ and μ	Nondimensional parameters in formulae for H_{01}, H_{02}, H_{11}, H_{12} and G_{01}, G_{02}, G_{11}, G_{12} respectively
ψ	Stream function
ρ, ϕ	Polar coordinates

Subscripts

i	At the point i of spline
i,j	At the intersection point of i and j splines
p and q	Components and projections along p and q respectively
p_o and q_o	Components and projections along p_o and q_o respectively
x, y, z	Projections along x, y, z respectively

Superscripts

i	Analytical expression along segment i of spline

i,j   Analytical expression for element formed by segments of i, i+1 and j, j+1 splines

1.0 INTRODUCTION

Results of flow analyses (based on a finite element analysis or a finite difference technique) in turbomachines are inevitably given in the form of a finite set of points (defined by Cartesian coordinates) describing the blade surface, and relative flow velocity components at these points. These velocity com-ponents form a relative velocity vector tangential to the blade surface. Thus the blade surface is a stream surface. However, it is impossible to apply these results directly to a three-dimensional boundary layer analysis of the flow along the surface. A widely accepted way of performing an analysis of this type requires the knowledge of a curvilinear coordinate system formed by the relative flow streamlines and perpendiculars to these streamlines, along with the velocity distribution expressed in this coordinate system. In the present work, the generation of the nonorthogonal curvilinear coordinate system (which is comprised of the streamlines and their corresponding orthogonals) is based on a new splining technique. The splining technique is applied to geometrical properties of curves, surfaces and the piecewise analytical presentation of functions of one and two variables. The splines presented in this work are used in the form of the Hermite Interpolant Polynomial [1] and the new approach is used in the determination of the first derivatives at the nodal points. The streamlines are computed using a method which involves a numerical solution of a system of ordinary differential equations developed by A. Gokhman and R. Goldstein [2].

2.0 SPLINING TECHNIQUE BASED ON THE HERMITE INTERPOLANT POLYNOMINAL

2.1 The Function of One Variable

The well known formula for the Hermite Interpolant Polynomial for the representation of a function of one variable, $y^i(x)$, on a segment $[x_i, x_{i+1}]$, is the following:

$$y^i(x) = y_i H_{01} + y_{i+1} H_{02} + \Delta x_i \left(\frac{dy}{dx}\Big|_i H_{11} + \frac{dy}{dx}\Big|_{i+1} H_{12} \right) \quad (1)$$

where $x_i \leq x \leq x_{i+1}$,

and

$$\begin{aligned} H_{01} &= 1 - 3\lambda^2 + 2\lambda^3 \\ H_{02} &= 3\lambda^2 - 2\lambda^3 \\ H_{11} &= \lambda - 2\lambda^2 + \lambda^3 \\ H_{12} &= \lambda^3 - \lambda^2 \end{aligned} \quad (2)$$

Here,

$$\lambda = \frac{x - x_i}{\Delta x_i}$$

and (3)

$$\Delta x_i = x_{i+1} - x_i$$

Previously, this formula has been utilized for interpolation of $y^i(x)$ in the case when $\frac{dy}{dx}\Big|_i$ and $\frac{dy}{dx}\Big|_{i+1}$ were known, or in the one-dimensional finite element analysis when $\frac{dy}{dx}\Big|_i$ and $\frac{dy}{dx}\Big|_{i+1}$ were considered as independent variables and found as a result of solution. However, to the best of the authors' knowledge, this polynomial has never been applied to the spline representation of a function when only the values of y_i are known at each nodal point along a curve representing the function $y = y(x)$.

Applying formula (1) to the segment $[x_{i+1}, x_{i+2}]$ one receives:

$$y^{i+1}(x) = y_{i+1} H_{01} + y_{i+2} H_{02} + \Delta x_i \left(\frac{dy}{dx}\Big|_{i+1} H_{11} + \frac{dy}{dx}\Big|_{i+2} H_{12} \right) \quad (4)$$

where

$$\lambda = \frac{x - x_{i+1}}{\Delta x_{i+1}} \quad (5)$$

Using (1), (2) and (4) it is easy to see that the Hermite Interpolant Polynomial assures the continuity of the functional values at each node since:

$$y^i(x_{i+1}) = y(x)^i\Big|_{\lambda=1} = y_{i+1}$$

$$y^{i+1}(x_{i+1}) = y^{i+1}(x)\Big|_{\lambda=0} = y_{i+1}$$

Using (1), the derivative of $y^i(x)$ by x can be obtained in this way:

$$\frac{dy^i(x)}{dx} = \frac{dy^i(x)}{d\lambda} \frac{d\lambda}{dx}$$

or using (3);

$$\frac{dy^i(x)}{dx} = \frac{dy^i(x)}{d\lambda} \frac{1}{\Delta x_i} \quad (6)$$

Finally;

$$\frac{dy^i(x)}{dx} = \left[y_i \frac{dH_{01}}{d\lambda} + y_{i+1} \frac{dH_{02}}{d\lambda} + \Delta x_{i+1} \left(\frac{dy}{dx}\Big|_i \frac{dH_{11}}{d\lambda} + \frac{dy}{dx}\Big|_{i+1} \frac{dH_{12}}{d\lambda} \right) \right] \frac{1}{\Delta x_i} \quad (7)$$

where:

$$\frac{dH_{01}}{d\lambda} = -6\lambda + 6\lambda^2$$

$$\frac{dH_{02}}{d\lambda} = 6\lambda - 6\lambda^2$$

$$\frac{dH_{11}}{d\lambda} = 1 - 4\lambda + 3\lambda^2 \qquad (8)$$

$$\frac{dH_{12}}{d\lambda} = 3\lambda^2 - 2\lambda$$

It can be seen from (3) and (6) that:

$$\frac{dy^i(x)}{dx}\Big|_{x=x_{i+1}} = \frac{dy^i(x)}{d\lambda}\Big|_{\lambda=1} \frac{1}{\Delta x_i}$$

and from (4) and (5) that:

$$\frac{dy^{i+1}(x)}{dx}\Big|_{x=x_{i+1}} = \frac{dy^{i+1}(x)}{d\lambda}\Big|_{\lambda=0} \frac{1}{\Delta x_{i+1}}$$

or, using (4), (7) and (8) one receives:

$$\frac{dy^i(x)}{dx}\Big|_{x=x_{i+1}} = \frac{dy^{i+1}(x)}{dx}\Big|_{x=x_{i+1}} = \frac{dy}{dx}\Big|_{i+1} \qquad (9)$$

Formula (9) illustrates that the Interpolant Polynomials (1) and (4), assure first derivative continuity at each nodal point.

If a function $y=y(x)$ is defined by values of y_i at x_i $(i=1,\ldots,n)$ points , determination of the values $\frac{dy}{dx}\big|_i$ $(i=2,\ldots,n-1)$ requires the imposition of the following conditions:

1. The continuity of the second derivatives at each nodal point $(i=2,\ldots n-1)$ must be satisfied.

2. Additional conditions at the points i=1 and i=n must be provided. These conditions could be:

   * The values of $\frac{dy}{dx}\big|_{i=1}$ and $\frac{dy}{dx}\big|_{i=n}$ are known a priori

   ** The second derivatives at i=1 and i=n are accepted as zero.

In this connection, it is necessary to note that when dealing with periodic functions these additional conditions are not needed. One can simply use equality of the first and second derivative at the points i=1 and i=n.

Using the chain rule and (7) one can obtain the following expression for the second derivative $\frac{d^2y^i(x)}{dx^2}$, on segment $[x_i, x_{i+1}]$:

$$\frac{d^2y^i(x)}{dx^2} = \frac{1}{\Delta x_i^2}\Big[y_i \frac{d^2H_{01}}{d\lambda^2} + y_{i+1}\frac{d^2H_{02}}{d\lambda^2} +$$

$$\Delta x_{i+1}\Big(\frac{dy}{dx}\Big|_i \frac{d^2H_{11}}{d\lambda^2} + \frac{dy}{dx}\Big|_{i+i} \frac{d^2H_{12}}{d\lambda^2} \Big) \Big] \qquad (10)$$

where

$$\frac{d^2H_{01}}{d\lambda^2} = -6 + 12\lambda$$

$$\frac{d^2H_{02}}{d\lambda^2} = 6 - 12\lambda$$

$$\frac{d^2H_{11}}{d\lambda^2} = -4 + 6\lambda \qquad (11)$$

$$\frac{d^2H_{12}}{d\lambda^2} = 6\lambda - 2$$

To obtain the condition necessary to guarantee continuity of the second derivative at the nodal points, an expression similar to (10) is developed for segment $[x_{i+1}, x_{i+2}]$. Thus, if $\frac{d^2y^i(x)}{dx^2}$ is equated to $\frac{d^2y^{i+1}(x)}{dx^2}$ at x_{i+1} one obtains:

$$\frac{1}{\Delta x_i^2}\Big[6y_i - 6y_{i+1} + \Delta x_i\Big(2\frac{dy}{dx}\Big|_i + 4\frac{dy}{dx}\Big|_{i+1}\Big)\Big]$$

$$= \frac{1}{\Delta x^2_{i+1}}\Big[-6y_{i+1} + 6y_{i+2} - \Delta x_{i+1}\Big(4\frac{dy}{dx}\Big|_{i+1} + 2\frac{dy}{dx}\Big|_{i+2}\Big)\Big] \qquad (12)$$

Therefore, solving for $\frac{dy}{dx}\big|_{i+1}$:

$$\frac{dy}{dx}\Big|_{i+1} = \frac{3\Big[\frac{1}{\Delta x_i^2}(-y_i + y_{i+1}) + \frac{1}{\Delta x^2}(-y_{i+1} + y_{i+2})\Big]}{2\Big(\frac{1}{\Delta x_i} + \frac{1}{\Delta x_{i+1}}\Big)} - \frac{\frac{dy}{dx}\big|_i \frac{1}{\Delta x_i} + \frac{dy}{dx}\big|_{i+2}\frac{1}{\Delta x_{i+1}}}{\frac{1}{\Delta x_i} + \frac{1}{\Delta x_{i+1}}} \qquad (13)$$

Formula (13) is thus used for computing the first derivative by a method of iterations. As previously mentioned, it is necessary to determine values for the derivatives only at the included points $(i=2,\ldots,n-1)$. Calculation of the first derivatives at the extreme points is obviously not necessary when specified. In the case when the second derivatives are accepted as zero at x_i and x_n $\frac{dy}{dx}\big|_1$ is expressed using $\frac{dy}{dx}\big|_2$ and $\frac{dy}{dx}\big|_n$ is expressed using $\frac{dy}{dx}\big|_{n-1}$. Indeed using (10) one can obtain:

$$\frac{dy}{dx}\Big|_1 = 0.5\ [3(y_2 - y_1)/\Delta x_1 - \frac{dy}{dx}\Big|_2]$$

and

$$\frac{dy}{dx}\Big|_n = 0.5\ [3(y_n - y_{n-1})/\Delta x_{n-1} - \frac{dy}{dx}\Big|_{n-1}]$$

As with any iteration scheme, an initial estimation must be used to begin the process. In this method an initial iteration for the first derivitive at each interior node is computed using the simplest finite difference formula. Consequent iterations using (13) illustrate expedient convergence. The comparison with conventional splining methods, which utilize the solution of a tri-diagonal matrix, shows substantial savings in computational time. As it is well known computation of the first derivative by this method yields high accuracy.

Evidently formula (1) yields the possibility of interpolation for values of x_o by a given y_o. Establishing to which segment x_o belongs is the first step (it may, of course, be on several segments). Second, on each segment i to which y_o belongs ($y_i \le y_o \le y_{i+1}$ or $y_{i+1} \le y_o \le y_i$) one must compute the first iteration for x_o^1 using linear interpolation. Later the following cubic equation must be solved:

$$y_o = y_i H_{01} + y_{i+1} H_{02} + \Delta x_i (\frac{dy}{dx}\Big|_i H_{11} + \frac{dy}{dx}\Big|_{i+1} H_{12}) \qquad (14)$$

Equation (14) is solved by the method of iterations presenting:

$$\lambda_o = \lambda_o^1 + \delta\lambda_o$$

where: $$\lambda_o^1 = \frac{x_o^1 - x_i}{\Delta x_i}$$

Using (14) one can obtain a first order equation for determing $\delta\lambda_o$:

$$A\delta\lambda_o + B + \varepsilon = 0 \qquad (15)$$

where: A and B are the second and third order polynomials of λ_o^1; and ε is the polynomial of two variables (λ_o^1 and $\delta\lambda_o$) which contains the higher order degrees of $\delta\lambda_o$.

The solution of (15) converges very quickly and yields the value of x_o in question as:

$$x_o = \lambda_o \Delta x_i + x_i$$

2.2 The Geometrical Curve

The same approach can also be applied to a piecewise representation of curves in parametrical form. A curve, given by a finite set of Cartesian coordinates, (x_i, y_i, z_i) i = 1,...,n can be presented in parametrical form as:

$$x = x(\ell) \quad y = y(\ell) \quad z = z(\ell) \qquad (16)$$

where ℓ is a parameter. It is very useful here to define ℓ as length along the curve since it will allow the computation of differential parameters at any point. Accepting this definition for ℓ it is then possible to represent coordinate x^i ($x_i \le x^i \le x_{i+1}$) using the Hermite Interpolant Polynomial:

$$x^i(\ell) = x_i H_{01} + x_{i+1} H_{02} + \Delta\ell_i (\frac{dx}{d\ell}\Big|_i H_{11} + \frac{dx}{d\ell}\Big|_{i+1} H_{12}) \qquad (17)$$

where H_{01}, H_{02}, H_{11} and H_{12} are defined by (2) and similarly, λ can be expressed as:

$$\lambda = \frac{\ell - \ell_i}{\Delta\ell_i} \qquad (18)$$

Formulae for y and z are similar to (18).

If it is desired to declare ℓ as the true length along the curve at nodal points (it can obviously not be equal to the true length at each point), one must also find the length by a method of iterations. Iterations for length and first derivatives for each node must be properly combined. At the outset, the parameter ℓ at each nodal point is determined using the Pythagorean theorem:

$$\Delta\ell_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 + (z_{i+1} - z_i)^2} \qquad (19)$$

Using the first iteration for length at the nodal points, the derivatives $\frac{dx}{d\ell}\Big|_i$, $\frac{dy}{d\ell}\Big|_i$ and $\frac{dz}{d\ell}\Big|_i$ (i=1,...,n) are determined in the same manner as before when dealing with a function of one variable. For subsequent iterations, $\Delta\ell_i$ is calculated using:

$$\Delta\ell_i = \int_0^1 [(\frac{dx^i(\lambda)}{d\lambda})^2 + (\frac{dy^i(\lambda)}{d\lambda})^2 + (\frac{dz^i(\lambda)}{d\lambda})^2]^{\frac{1}{2}} d\lambda \qquad (20)$$

where:

$$\frac{dx^i(\ell)}{d\lambda} = x_i \frac{dH_{01}}{d\lambda} + x_{i+1}\frac{dH_{02}}{d\lambda} + \Delta\ell_i (\frac{dx}{d\ell}\Big|_i \frac{dH_{11}}{d\lambda} + \frac{dx}{d\ell}\Big|_{i+1}\frac{dH_{12}}{d\lambda}) \qquad (21)$$

Formulae for $\frac{dy^i(\ell)}{d\lambda}$ and $\frac{dz^i(\ell)}{d\lambda}$ are similar.

Both iterations converge very quickly and if the proper number and distribution of points along the curve are used, results of the first derivative computation are very accurate. Illustration of this can be found in Table 1 which presents a comparison of the first and second derivatives computed by the proposed method, with exact analytical values for the curve $y = \sin x$ $[0 \leq x \leq 2\pi]$. As seen in Table 1, $(\frac{dx}{d\ell})^2 + (\frac{dy}{d\ell})^2$ is not exactly equal to one. This slight inequality is attributed to the fact that the parameter ℓ is equal to genuine length along the curve only at nodal points. Table 1 also shows a close apoproximation between calculated and analytical values of the second derivative. Thus, the proposed technique can be successfully applied (especially when the points are not near the ends and where second derivatives are accepted as zero) for computing second derivatives and subsequently curvature and torsion. Table 2 shows a comparison of differential parameters and length at the nodal points computed by the proposed method with analytically obtained values for a cardioid. $\rho = a(1-\cos\phi)$ $[0\leq\phi\leq 2\pi]$

It is necessary to mention that in the case of a periodic or closed curve, with continuous first and second derivatives at each point (cardioid has a discontinuity of the first derivative at the point $\phi=0$), conditions 2 (Section 2.1) at the ends are not required.

The system of equations presented by (17) allows interpolation, i.e., with one given coordinate it is possible to find the reamining two (for example, if x_o is given, y_o and z_o may be determined). One must first find all segments to which x_o belongs. Secondly, one has to find the first iteration for y_o and z_o using linear interpolation:

$$y_o^1 = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x_o - x_i)$$

and similarly:

$$z_o^1 = \frac{z_{i+1} - z_i}{x_{i+1} - x_i}(x_o - x_i)$$

Now, the first iteration for λ can be established as:

$$\lambda_o^1 = \frac{[(x_o - x_i)^2 + (y_o^1 - y_i)^2 + (z_o^1 - z_i)^2]^{\frac{1}{2}}}{\Delta\ell_i}$$

Knowing λ_o^1 one can easily solve the equation:

$$x_o = f(\lambda_o^1, \delta\lambda)$$

by the method described for the function of one variable.

3.0 APPLICATION OF SPLINING TECHNIQUE TO FUNCTION OF TWO VARIABLES AND GEOMETRICAL SURFACE

3.1 Function of Two Variables

If a function of two variables $x = f(x_i, y_j)$ is defined by a set of values $z_{i,j}$ (x,y) on each rectangle (with vertices (x_i,y_i), (x_{i+1},y_j), (x_{i+1},y_{j+1}) and (x_i,y_{j+1})) and as:

TABLE 1

Splining Results for Line y = sin x

x	y	dx/dℓ	dy/dℓ	δ(dx/dℓ)	δ(dy/dℓ)	δ[(dx/dℓ)² + (dy/dℓ)²]	δ(d²x/dℓ²)	δ(d²y/dℓ²)
0.00000	0.00000	0.70710	0.70712	0.00001	0.00001	0.00000	0.00000	0.00000
0.58905	0.01028	0.76892	0.63938	0.00000	0.00004	0.00002	0.00027	0.00312
1.37445	0.02399	0.98144	0.19129	0.00006	0.00019	0.00009	0.00880	0.00928
2.15984	0.03769	0.87421	-0.48564	0.00006	0.00002	0.00005	0.00366	0.00560
2.94524	0.05138	0.71393	-0.70023	0.00001	0.00002	0.00001	0.00040	0.00091
3.73064	0.06507	0.76892	-0.63938	0.00000	0.00004	0.00002	0.00027	0.00312
4.51604	0.07874	0.98144	-0.19129	0.00006	0.00019	0.00009	0.00880	0.00928
5.30144	0.09240	0.87421	0.48564	0.00006	0.00002	0.00005	0.00366	0.00560
6.28319	0.10944	0.70710	0.70712	0.00001	0.00002	0.00001	0.00000	0.00000

For any function F presented in Table 1:

$\delta(F) = |F - F_e|$

where F and F_e are the computed and exact values of F respectively.

TABLE 2

Splining Results for Cardioid $\rho = a(1-\cos\phi)$ [a = 1.0]
Using 33 Points

ϕ	ρ	$dx/d\ell$	$dy/d\ell$	$\delta(dx/d\ell)$	$\delta(dy/d\ell)$	$\delta[(dx/d\ell)^2 + (dy/d\ell)^2]$	ℓ	$\delta(\ell)$
0.00000	0.00000	1.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.58905	0.16853	0.63389	0.77532	0.00050	0.00231	0.00147	0.17227	0.00016
1.37445	0.80491	-0.47139	0.88197	0.00000	0.00005	0.00004	0.90798	0.00003
2.15984	1.55557	-0.99518	-0.09800	0.00000	0.00002	0.00000	2.11443	0.00001
2.94524	1.98079	-0.29028	-0.95693	0.00000	0.00001	0.00001	3.60793	0.00000
3.73064	1.83147	0.77301	-0.63438	0.00000	0.00002	0.00001	5.16113	0.00000
4.51604	1.19509	0.88192	0.47142	0.00001	0.00003	0.00001	6.53757	0.00000
5.30144	0.44443	-0.09797	0.99543	0.00004	0.00024	0.00024	7.52766	0.00000
6.28319	0.00000	-1.00000	0.00000	0.00000	0.00000	0.00000	8.00000	0.00000

For any function F presented in Table 2:

$$\delta(F) = |F - F_e|$$

where F and F_e are the computed and exact values of F respectively.

and

$$\delta(\ell) = \frac{|\ell-\ell_e|}{\ell_e} \quad \text{for } \ell>0$$

where ℓ and ℓ_e are the computed and exact values of ℓ respectively.

$$z^{i,j} = z_{i,j} H_{01} G_{01} + z_{i,j+1} H_{02} G_{01} + z_{i+1,j} H_{01} G_{02} + z_{i+1,j+1} H_{02} G_{02}$$
$$+ \left(\frac{dz}{dx}\Big|_{i,j} H_{11} + \frac{dz}{dx}\Big|_{i,j+1} H_{12} \right) \Delta x_j G_{01}$$
$$+ \left(\frac{dz}{dx}\Big|_{i+1,j} H_{11} + \frac{dz}{dx}\Big|_{i+1,j+1} H_{12} \right) \Delta x_j G_{02}$$
$$+ \left(\frac{dz}{dy}\Big|_{i,j} G_{11} + \frac{dz}{dy}\Big|_{i+1,j} G_{12} \right) \Delta y_i H_{01}$$
$$+ \left(\frac{dz}{dy}\Big|_{i,j+1} G_{11} + \frac{dz}{dy}\Big|_{i+1,j+1} G_{12} \right) \Delta y_i H_{02} \qquad (22)$$

where: $\lambda = \frac{x-x_j}{\Delta x_j} \quad \mu = \frac{y-y_i}{\Delta y_i}$

The values of the derivatives $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at each nodal point can be computed by successive application of the splining technique, for the function of one variable, along lines defined by $x = x_i$ and $y = y_j$.

Formula (22) clearly develops a piecewise representation of the function of two variables that gives continuity of the function and its partial derivatives at each point of the domain, including the boundary points of each rectangle.

Assessment of the accuracy in presenting a function of two variables was performed using an ellipsoid:

$$z = c \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{0.5} \qquad (23)$$

Results of this comparison (Table 3) show excellent correlation between the function $z = z(x_i, y_j)$ and the partial derivatives $\frac{dz}{dx}$ and $\frac{dz}{dy}$.

Geometrical Surfaces

An arbitrary geometrical surface, defined as a set of points $x_{i,j}$, $y_{i,j}$, $z_{i,j}$ (i=1,...,n;j=1,...,m), can be formed by two intersecting systems of curves which lay on the surface. These two systems of curves, formed by applying the splining technique to each system, do not have to be orthogonal. Let ℓ be the length along the first set of curves formed by the points $x_{i,j}$, $y_{i,j}$, $z_{i,j}$ (i=1,...,n;j = constant), and s be the length along the second set of curves formed by the points $x_{i,j}$, $y_{i,j}$, $z_{i,j}$ (i= constant, j=1,...,m). As a result of computing the splines

TABLE 3

Splining Results for for a Function of Two Variables

$$z = c\,(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})^{\frac{1}{2}}$$

49 x 49 Points

x	y	z	dz/dx	dz/dy	δ(dz/dx)	δ(dz/dy)
-3.0000	-1.8000	0.5568	1.3470	1.4368	0.0000	0.0000
-1.3200	-1.8000	1.4575	0.2264	0.5489	0.0000	0.0000
0.4800	-1.8000	1.5819	-0.0759	0.5057	0.0000	0.0000
2.2800	-1.8000	1.1227	-0.5077	0.7126	0.0000	0.0000
-3.0000	-0.0720	1.3220	0.5673	0.0242	0.0000	0.0000
-1.3200	-0.0720	1.8874	0.1748	0.0170	0.0000	0.0000
0.4800	-0.0720	1.9850	-0.0605	0.0161	0.0000	0.0000
2.2800	-0.0720	1.6426	-0.3470	0.0195	0.0000	0.0000
-3.0000	1.6560	0.7288	1.0291	-1.0098	0.0000	0.0000
-1.3200	1.6560	1.5315	0.2155	-0.4806	0.0000	0.0000
0.4800	1.6560	1.6503	-0.0727	-0.4460	0.0000	0.0000
2.2800	1.6560	1.2172	-0.4683	-0.6047	0.0000	0.0000

For any function F presented in Table 3:

$\delta(F) = |F - F_e|$

where F and F_e are the computed and exact values of F respectively.

along these systems of curves one obtains a set of values $\ell_{i,j}$ and $s_{i,j}$ (i=1,...,n;j=1,...,m). Now, the coordinates x,y,z of an arbitrary quadrilateral on the surface, $(x_{i,j}, y_{i,j}, z_{i,j})$, $(x_{i+1,j}, y_{i+1,j}, z_{i+1,j})$ $(x_{i+1,j+1}, y_{i+1,j+1}, z_{i+1,j+1})$ and $(x_{i,j+1}, y_{i,j+1}, z_{i,j+1})$, can be expressed using the splining technique. The formula for an x coordinate, expressed in this way is the following:

$$x^{i,j} = x_{i,j} H_{01} G_{01} + x_{i,j+1} H_{02} G_{01} + x_{i+1,j} H_{01} G_{02} + x_{i+1,j+1} H_{02} G_{02}$$

$$+ \left(\frac{dx}{d\ell}\Big|_{i,j} H_{11} + \frac{dx}{d\ell}\Big|_{i,j+1} H_{12} \right) \Delta\ell_{i,j} G_{01}$$

$$+ \left(\frac{dx}{d\ell}\Big|_{i+1,j} H_{11} + \frac{dx}{d\ell}\Big|_{i+1,j+1} H_{12} \right) \Delta\ell_{i+1,j} G_{02}$$

$$+ \left(\frac{dx}{ds}\Big|_{i,j} G_{11} + \frac{dx}{ds}\Big|_{i+1,j} G_{12} \right) \Delta s_{i,j} H_{01}$$

$$+ \left(\frac{dx}{ds}\Big|_{i,j+1} G_{11} + \frac{dx}{ds}\Big|_{i+1,j+1} G_{12} \right) \Delta s_{i,j+1} H_{02} \qquad (24)$$

The formulae for y and z are similar.

The H and G coefficients are defined by equations similar to (3), yet are now distinguished by usage of the independent parameters λ and μ which may vary from 0 to 1.

Values of $\Delta\ell_{i,j}, .., \Delta s_{i+1,j+1}$ and $\frac{dx}{d\ell}\Big|_{i,j}, .., \frac{dx}{ds}\Big|_{i+1,j+1}$ were determined during computation of the i and j splines.

Since (23) presents the surface passing through curves formed by the system of splines, this surface is continuous at every point. However, due to the fact that $\Delta\ell$ and Δs depend on a system of two indices (when dealing with a function of two variables Δx changes only with i, and Δy only with j, thus assuring continuity of the partial derivatives by x and y on the entire element boundaries) on the boundary of the quadrilateral the partial derivatives:

$$\frac{dx^{i,j}}{d\ell}, \frac{dy^{i,j}}{d\ell}, \frac{dz^{i,j}}{d\ell}, \frac{dx^{i,j}}{ds}, \frac{dy^{i,j}}{ds} \text{ and } \frac{dz^{i,j}}{ds}$$

are continuous (in both the i and j directions) only at the nodal points. At all other points on the boundary the partial derivatives are discontinuious in one direction, i.e., $\frac{dx^{i,j}}{d\ell}$, $\frac{dy^{i,j}}{d\ell}$ and $\frac{dz^{i,j}}{d\ell}$ are continuous along the i splines and $\frac{dx^{i,j}}{ds}$, $\frac{dy^{i,j}}{ds}$ and $\frac{dz^{i,j}}{ds}$ are continuous along the j splines. Evidently, all respective partial derivatives are continuous at all interior points of a quadrilateral.

Assessment of the accuracy of representing a geometrical surface was evaluated using a two axis ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

In order to produce this evaluation, the systems of i and j lines were accepted as splines passing through the points of intersection of the parallels and meridonals of the ellipsoid. Results of this comparison show a high accuracy in presenting the coordinates yet a slightly decreased accuracy in the presentation of the partial derivatives (Table 4). The larger inaccuracies in predicting the partial derivatives, in comparison with those when dealing with the function of two variables, can be associated with the discontinuity along the boundaries.

Use of equations (24) yields the possibility of interpolation for the third coordinate if the other two are given. For example, consider that x_o and y_o are specified and that z_o must be found. First, one must identify all elements whose projections on plane XOY comprise the point x_o, y_o as an inside point. The criteria for determining whether this point x_o, y_o is an interior point of a quadrilateral is the equality of the viewing angle (α) to 2π (Figure 1). For an outside point α must be equal to zero. The following formula is used to compute α:

$$\alpha = \oint_{c_i} \frac{(x-x_o)dy - (y-y_o)dx}{(x-x_o)^2 + (y-y_o)^2} \qquad (25)$$

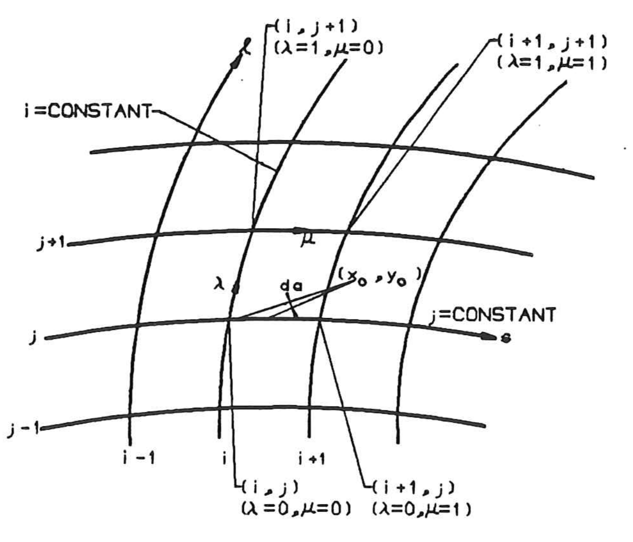

Fig. 1 The element on stream surface

where: c_i is the counter of the quadrilateral element.

Proof of this formula for α can be obtained through introduction of the polar coordinate system:

$$x-x_o = R\cos\alpha$$
$$y-y_o = R\sin\alpha$$

Once all elements which contain x_o, y_o are identified, one has to find the proper values of λ and μ for each designated element. This operation involves the solution of two sixth order equations. This solution can be found by the method of iterations using linearized equations for each iteration. A detailed description of this solution exceeds the scope of the present paper.

4.0 NUMERICAL METHOD OF STREAMLINE CALCULATION IN 3-D SPACE FOR A SPECIFIED VELOCITY FIELD

As previously mentioned the streamline computation is based on the Gokhman-Goldstein method [2].

Let ℓ be length along the streamline and $\vec{V}$ be velocity, then the differential equation of a streamline in three-dimensional space in vector form is

$$\vec{V} \uparrow\uparrow d\vec{\ell} \qquad (26)$$

where

$d\vec{\ell} = d\ell\hat{\tau}$ ($\hat{\tau}$ is the unit vector tangential to streamline)

But in scalar form

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$d\vec{\ell} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Therefore, the equation (26) gives

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z} = \frac{d\ell}{V} \qquad (27)$$

where

$$V = (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}}$$

The equations (27) can be solved by different numerical methods providing that the velocity field at each point is known.

The reason for accepting the Gokhman-Goldstein method is that it yields a very accurate solution with relatively small computational time involved.

The unknown values x, y, and z which are Cartesian coordinates of points along the streamline being calculated, can be presented as functions of length ℓ along the streamline by:

$$x = x(\ell)$$
$$y = y(\ell) \qquad (28)$$
$$z = z(\ell)$$

Now the equations (27) can be rewritten as a system of three ordinary differential equations of the first order:

TABLE 4

Application of Splining Technique to Ellipsoid Surface

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

(Element with Indices i = 6,j = 7)

λ	μ	x	y	z	δ(z)	dz/dx	dz/dy	δ(dz/dx)	δ(dz/dy)
0.00000	0.60000	1.15176	-1.74084	1.15176	0.00001	-0.99998	0.67172	0.00002	0.00005
0.10000	0.40000	1.12640	-1.78229	1.14867	0.00001	-0.98036	0.68957	0.00025	0.00004
0.20000	0.60000	1.12890	-1.74084	1.17417	0.00002	-0.96156	0.65909	0.00011	0.00015
0.30000	0.30000	1.09662	-1.80290	1.16308	0.00003	-0.94246	0.68896	0.00039	0.00002
0.40000	0.70000	1.11229	-1.72002	1.20332	0.00004	-0.92450	0.63550	0.00013	0.00020
0.50000	0.10000	1.05946	-1.84390	1.16894	0.00004	-0.90635	0.70074	0.00001	0.00032
0.60000	0.00000	1.04084	-1.86429	1.17130	0.00000	-0.88863	0.70673	0.00001	0.00067
0.70000	0.50000	1.06340	-1.76160	1.22064	0.00001	-0.87127	0.64169	0.00008	0.00028
0.80000	0.20000	1.03135	-1.82344	1.20771	0.00001	-0.85443	0.67121	0.00046	0.00018
0.90000	0.90000	1.06427	-1.67817	1.27109	0.00001	-0.83711	0.58665	0.00018	0.00013
1.00000	1.00000	1.05765	-1.65714	1.28875	0.00000	-0.82068	0.57146	0.00000	0.00003

For any function F presented in Table 4:

$$\delta(F) = |F - F_e|$$

where F and F_e are the computed and exact values of F respectively.

and

$$\delta(z) = \frac{|z-z_e|}{D_e}$$

where z and z_e are the computed and exact values of z respectively

D_e is the maximum diameter of the element.

$$\frac{dx}{d\ell} = \frac{V_x}{V}$$

$$\frac{dy}{d\ell} = \frac{V_y}{V} \qquad (29)$$

$$\frac{dz}{d\ell} = \frac{V_z}{V}$$

If the initial point of the streamline has coordinates x_o, y_o and z_o, the functions $x = x(\ell)$, $y = y(\ell)$ and $z = z(\ell)$ can be expressed by means of Taylor's a series:

$$x = x_o + \sum_{i=1}^{N} \frac{d^{(i)}x}{d\ell^{(i)}}\Big|_o \frac{\ell^i}{i!}$$

$$y = y_o + \sum_{i=1}^{N} \frac{d^{(i)}y}{d\ell^{(i)}}\Big|_o \frac{\ell^i}{i!} \qquad (30)$$

$$z = z_o + \sum_{i=1}^{N} \frac{d^{(i)}z}{d\ell^{(i)}}\Big|_o \frac{\ell^i}{i!}$$

where;

N is the highest order derivative being considered, and

ℓ is equal to zero at the point $x = x_o$, $y = y_o$, $z = z_o$

Let us accept the value ΔL for the streamline computation of the segment $[0 \le \ell \le \Delta L]$. Streamline computation is then conducted by the method of interations.

For a first iteration of the streamline on segment $(0 \le \ell \le \Delta L)$ only the first derivatives in (30) are used since they are known a priori. So for the first interation:

$$x' = x_o + \frac{dx}{d\ell}\Big|_o \ell$$

$$y' = y_o + \frac{dy}{d\ell}\Big|_o \ell$$

$$z' = z_o + \frac{dz}{d\ell}\Big|_o \ell$$

The values of the higher order derivatives at the point, x_o, y_o, z_o $\left(\frac{d^2x}{d\ell^2}\Big|_o, \frac{d^2y}{d\ell^2}\Big|_o, \frac{d^2z}{d\ell^2}\Big|_o, \ldots, \frac{d^{(N)}z}{d\ell^{(N)}}\Big|_o\right)$ are unknown and have to be determined. In order to determine these higher order derivatives the segment $(0 \le \ell \le \Delta L)$ is divided into N-1 segments by these values of ℓ

$$\ell_k = \frac{\Delta L}{N-1} k \qquad (1 \le k \le N-1)$$

Now the values of x, y and z are calculated by (30) at the points ℓ_i and after that the values of V_x, V_y, V_z and V are also calculated at the points ℓ_i (the values of V_x, V_y and V_z can be calculated when values of x, y and z are known, as stated above).

The values of $\frac{dx}{d\ell}$, $\frac{dy}{d\ell}$, $\frac{dz}{d\ell}$ can be also introduced using Taylor's series by differentiation of formulae (30).

$$\frac{dx}{d\ell} = \frac{dx}{d\ell}\Big|_o + \sum_{i=2}^{N} \frac{d^{(i)}x}{d\ell^{(i)}}\Big|_o \frac{\ell^{i-1}}{(i-1)!}$$

$$\frac{dy}{d\ell} = \frac{dy}{d\ell}\Big|_o + \sum_{i=2}^{N} \frac{d^{(i)}y}{d\ell^{(i)}}\Big|_o \frac{\ell^{i-1}}{(i-1)!} \qquad (31)$$

$$\frac{dz}{d\ell} = \frac{dz}{d\ell}\Big|_o + \sum_{i=2}^{N} \frac{d^{(i)}z}{d\ell^{(i)}}\Big|_o \frac{\ell^{i-1}}{(i-1)!}$$

On the other hand, the values of $\frac{dx}{d\ell}$, $\frac{dy}{d\ell}$ and $\frac{dz}{d\ell}$ are defined by values of V_x, V_y, V_z and V (see formulae (29)).

Therefore, one obtaines three independent systems of linear algebraic equations for determination of the first iteration

for the higher order derivatives $\left(\frac{d^2x}{d\ell^2}\Big|_o, \ldots, \frac{d^{(N)}z}{d\ell^{(N)}}\Big|_o\right)$ at the point x_o, y_o, z_o.

$$\left\{ \frac{dx}{d\ell}\Big|_k = \frac{dx}{d\ell}\Big|_o + \sum_{1=2}^{N} \frac{d^{(i)}x}{d\ell^{(i)}}\Big|_o \frac{\ell_k^{i-1}}{(i-1)!}\right.$$

$$\left\{ \frac{dy}{d\ell}\Big|_k = \frac{dy}{d\ell}\Big|_o + \sum_{i=2}^{N} \frac{d^{(i)}y}{d\ell^{(i)}}\Big|_o \frac{\ell_k^{i-1}}{(i-1)!}\right. \qquad (32)$$

$$\left\{ \frac{dz}{d\ell}\Big|_k = \frac{dz}{d\ell}\Big|_o + \sum_{i=2}^{N} \frac{d^{(i)}z}{d\ell^{(i)}}\Big|_o \frac{\ell_k^{i-1}}{(i-1)!}\right.$$

$(k = 1, 2, \ldots, N-2, N-1)$

After determining of the first iteration for the higher order derivatives $\left(\frac{d^2x}{d\ell^2}\Big|_o, \ldots, \frac{d^{(N)}z}{d\ell^{(N)}}\Big|_o\right)$ the second iteration of the coordinates (x, y, z) for the values ℓ_k is calculated by formulae (30) and after this the second iteration for $\frac{dx}{d\ell}$, $\frac{dy}{d\ell}$ and $\frac{dz}{d\ell}$ for the same values ℓ_k is calculated and so on, until iterations converge.

Convergence of these iterations was mathematically proven for a class of functions $V_x = V_x(x, y, z)$, $V_y = V_y(x, y, z)$ and $V_z = V_z(x, y, z)$ satisfiying the Lipschitz condition [2] so this method can be safely applied to a wide class of problems considered in fluid mechanics.

After determination of the streamline geometry for segment $(0 \le \ell \le L)$ the last point of the segment x_{N-1}, y_{N-1}, z_{N-1} is accepted as the initial point for the calculations of the second streamline segment with same length ΔL, until the streamline passes throughout the desired area of domain.

The accuracy of computations was evaluated by means of comparison with the theoretical solution for uniform flow passing around a circular cylinder.

The theoretical solution for flow around a circular cylinder is easily obtainable by usage of complex variable analysis. The complex potential for uniform flow (with velocity U) around dipole M is

$$W = \frac{M}{\pi}\frac{1}{z} + Uz \qquad (33)$$

Using (33) it is possible to obtain [3] the equation for a streamline passing (of flow around a cylinder) through the point x_o, y_o from the equation:

TABLE 5

Streamline Calculations Using Gokhman-Goldstein Method

x	y	δ(y)
-3.0000	0.2000	0.0000
-2.0008	0.2360	0.0000
-1.5552	0.2960	0.0001
-1.1158	0.5218	0.0012
-0.7381	0.8488	0.0006
-0.2872	1.0581	0.0001
0.2089	1.0746	0.0000
0.6714	0.8932	0.0004
1.0564	0.5756	0.0013
1.4778	0.3164	0.0002
2.9699	0.2005	0.0000
6.4698	0.1822	0.0000

$$\delta(y) = \frac{|y - y_e|}{a}$$

Where y and y_e are the computed and exact values respectively, a is the radius of the cylinder (a = 1.0)

$$\psi_o = Uy\left(1-\frac{a^2}{x^2+y^2}\right) \tag{34}$$

where

$$\psi_o = Uy_o\left(1-\frac{a^2}{x_o^2+y_o^2}\right)$$

a is a radius of cylinder.

The results of streamline calculation using the Gokhman-Goldstein method, as compared with exact analytical results, show a very high accuracy (Table 5).

5.0 COMPUTATION OF ORTHOGONAL CURVILINEAR COORDINATE SYSTEM BASED ON STREAMLINES

5.1 Generation of the Continuous Velocity Field in the Auxilliary Non-Orthogonal Curvilinear Coordinate System

Given the geometry of the surface as the set of points:

$$x_{i,j},\ y_{i,j},\ z_{i,j} \qquad (i=1,\dots,n;j=1,\dots,m) \tag{35}$$

and the velocity distribution:

$$V_{x_{i,j}},\ V_{y_{i,j}},\ V_{z_{i,j}} \qquad (i=1,\dots,n;j=1,\dots,m) \tag{36}$$

the results presented in Section 1.2 are applied by passing two systems of splines through the points describing the surface. This results in an auxilliary coordinate system p,q. In this coordinate system the lines p = constant represent j splines and lines q = constant represent i splines (see Section 1.2). The value of p is defined to be equal to length along one of the i splines and q equal to length along one of the j splines. Analytical expressions for p and q using λ and μ:

$$p = p(\lambda,\mu)$$
$$q = q(\lambda,\mu) \tag{37}$$

can be achieved using (22) as explained in 3.1.

As the next step in this method, projections (V_p and V_q) of the velocity vector ($\vec{V}$), along lines p and q, are calculated at each nodal point. The formulae for V_p and V_q can be obtained in the following way. Components of $\vec{V}$ along p and q at a point i,j are (in the following formulae, (37)-(42) the indices i and j are omitted for clarity of presentation):

$$\vec{V}_p = V_p \cdot \hat{e}_p \tag{38}$$

where $\hat{e}_p = \frac{dx}{d\ell}\hat{\imath} + \frac{dy}{d\ell}\hat{\jmath} + \frac{dz}{d\ell}\hat{k}$

$$\vec{V}_q = V_q \cdot \hat{e}_q \tag{39}$$

where $\hat{e}_q = \frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath} + \frac{dz}{ds}\hat{k}$

Values of the derivatives in (37) and (38) were determined while generating the coordinate system p,q. Next, it is evident that:

$$\vec{V} = \vec{V}_p + \vec{V}_q$$

or using (37) and (38)

$$\vec{V} = \left(V_p\frac{dx}{d\ell} + V_q\frac{dx}{ds}\right)\hat{\imath} + \left(V_p\frac{dy}{d\ell} + V_q\frac{dy}{ds}\right)\hat{\jmath} + \left(V_p\frac{dz}{d\ell} + V_q\frac{dz}{ds}\right)\hat{k} \tag{40}$$

therefore;

$$V_x = V_p\frac{dx}{d\ell} + V_q\frac{dx}{ds}$$
$$V_y = V_p\frac{dy}{d\ell} + V_q\frac{dy}{ds} \tag{41}$$

and finally:

$$V_p = \frac{V_x\frac{dy}{ds} - V_y\frac{dx}{ds}}{D}$$

$$V_q = \frac{V_y\frac{dx}{d\ell} - V_x\frac{dy}{d\ell}}{D}$$

where: $D = \frac{dx}{d\ell}\frac{dy}{ds} - \frac{dy}{d\ell}\frac{dx}{ds}$

Of course, instead of $V_{x_{i,j}}$ and $V_{y_{i,j}}$ projections, one can use $V_{x_{i,j}}$ and $V_{z_{i,j}}$. These computations will be equivalent since vector $\vec{V}$ is tangential to the plane formed by $\hat{e}_p$ and $\hat{e}_q$. Therefore, in the case where one of the velocity components is zero, the remaining two must be used in (41). Now, having the values $V_{p_{i,j}}$ and $V_{q_{i,j}}$ one can present (using the results of Section (3.1)) the velocity components, V_p and V_q, along the surface, as continuous differentiable functions of two variables (see Formula (22)):

$$V_p = V_p(p,q)$$
$$V_q = V_q(p,q)$$

Therefore, the values V_p and V_q can be calculated for any pair of p and q.

5.2 Computation of Streamlines and Perpendiculars

At this point one can calculate the streamlines using the method presented in Section 4.0. Streamlines are now calculated in the p,q coordinate system. A solution for each streamline was sought in the following form:

$$p = p(\ell_s)$$
$$q = q(\ell_s) \tag{42}$$

where ℓ_s is length along the streamline.

Values of $\frac{dp}{d\ell_s}$ and $\frac{dq}{d\ell_s}$, which are necessary for streamline calculations, are computed using the following formula:

$$\frac{dp}{d\ell_s} = \frac{V_p}{H_p V}$$

$$\frac{dq}{d\ell_s} = \frac{V_q}{H_q V} \qquad (43)$$

Formulae (42) follow from the definition of the streamline (See Figure 2)

$$\vec{d\ell} \uparrow\uparrow \vec{V}$$

where: $\vec{d\ell} = H_p dp\hat{e}_p + H_q dq\hat{e}_q$ (the differential element along streamline)

$\vec{V} = V_p\hat{e}_p + V_q\hat{e}_q$ (velocity vector)

After computation of the streamlines, the new orthogonal curvilinear coordinate system is devised where the streamlines are accepted as p_o coordinate

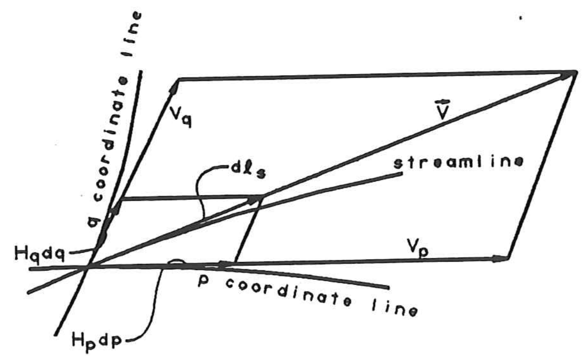

Fig. 2 Streamline on stream surface

TABLE 6

Results of Computation of Streamlines

x_s	y_s	z_s	$\delta(y_s,z_s)$	V_s	$\delta(V_s)$
-0.3943	0.4226	0.8161	0.0000	1.3784	0.0001
-0.2669	0.4431	0.8558	0.0017	1.4456	0.0000
-0.1199	0.4564	0.8817	0.0021	1.4892	0.0001
0.0299	0.4595	0.8877	0.0022	1.4993	0.0000
0.1790	0.4523	0.8737	0.0022	1.4578	0.0000
0.3241	0.4350	0.8401	0.0007	1.4191	0.0000
0.4619	0.4079	0.7876	0.0007	1.3305	0.0001
-0.3943	0.1226	0.9108	0.0000	1.3785	0.0000
-0.2670	0.1286	0.9551	0.0000	1.4455	0.0000
-0.1200	0.1325	0.9839	0.0001	1.4892	0.0000
0.0297	0.1334	0.9906	0.0002	1.4993	0.0000
0.1788	0.1313	0.9751	0.0001	1.4758	0.0000
0.3238	0.1262	0.9377	0.0000	1.4192	0.0000
0.4616	0.1184	0.8792	0.0000	1.3307	0.0000
0.5889	0.1078	0.8010	0.0002	1.2122	0.0000
0.7031	0.0949	0.7047	0.0001	1.0667	0.0002
0.8014	0.0797	0.5927	0.0018	0.8979	0.0007

$$\delta(V_s) = |V_s - V_{s_e}|/U$$

Where V_s and V_{s_e} are the computed and exact values respectively

U is velocity of uniform flow in infinity (U = 1.0).

$$\delta(y_s,z_s) = \frac{|y_s - y_{s_e}| + |z_s - z_{s_e}|}{D_e}$$

Where y_s and z_s are calculated values and y_{s_e} and z_{s_e} are exact values respectively

D_e is the diameter of element.

lines. Length along one of the streamlines is equated to the value of p_o at each point, and this streamline is divided into an accepted number of segments (not necessarily of equal length). Perpendiculars to the streamlines are passed through these segment division points and the system of q_o lines result. The value of q_o is assigned to the value of length along one of the perpendiculars. These perpendiculars are calculated using the same method as that used for the streamlines employing an auxilliary velocity field $\vec{V}^*$, which is perpendicular to the given field $\vec{V}$ (a suggestion made by an authors' colleague Mr. Gary Franke). At the conclusion of this process, intersections between the coordinate lines p_o and q_o (nodal points), are found and subsequently, values of Lamme coefficients (H_{p_o} and H_{q_o}) and the derivatives ($\frac{dV}{dp_o}$ and $\frac{dV}{dq_o}$) are determined for the new coordinate system. Also, the primary cartesian coordinates x,y and z for each nodal point are computed.

Accuracy of the solution was assessed by comparison with flow around a sphere [3] when flow in infinity is directed along the x-axis. Velocity components for this flow are computed using the following formulae:

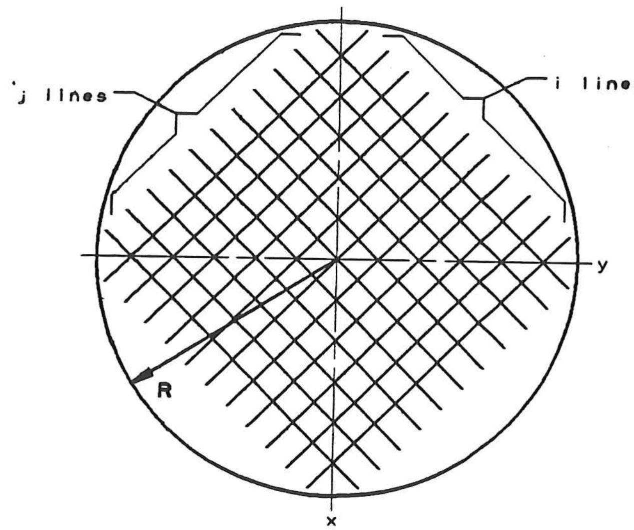

Fig. 3 The i and j lines system of streamline computation for the flow around a sphere

TABLE 7

Parameters In Orthogonal Curvilinear Coordinate System

p_o	q_o	x	y	z	V	H_{p_o}	H_{q_o}	dV/dp_o	dV/dq_o
.13111	.80349	-.35244	.43026	.83107	1.40373	1.00000	1.00000	.51136	-.00060
.16648	.83851	-.30997	.43710	.84431	1.42612	1.00000	1.01576	.46978	-.00085
.20177	.87355	-.26688	.44307	.85584	1.44557	1.00000	1.02973	.39877	-.00044
.23702	.90858	-.22325	.44812	.86565	1.46205	1.00000	1.04127	.33030	.00069
.27225	.94362	-.17914	.45226	.87371	1.47575	1.00000	1.05027	.29124	-.00109
.22830	.73147	-.35249	.32853	.87625	1.40369	1.00069	1.00000	.51077	.00011
.26149	.76526	-.30996	.33378	.89023	1.42606	1.00069	1.01582	.46957	.00005
.29475	.79898	-.26687	.33833	.90239	1.44554	1.00069	1.02965	.40096	.00010
.32808	.83271	-.22323	.34221	.91272	1.46212	1.00069	1.04134	.33004	.00032
.36147	.86643	-.17916	.34540	.92119	1.47567	1.00069	1.05101	.28651	-.00004
.31668	.65929	-.35244	.22668	.90797	1.40373	1.00027	1.00000	.51087	.00042
.34871	.69204	-.30995	.23031	.92244	1.42611	1.00027	1.01601	.46951	.00043
.38087	.72480	-.26684	.23345	.93504	1.44558	1.00027	1.02984	.40034	.00029
.41313	.75755	-.22321	.23613	.94574	1.45213	1.00027	1.04162	.33014	.00007
.44548	.79027	-.17916	.23832	.95452	1.47572	1.00027	1.05147	.28772	.00047
.39907	.58630	-.35244	.12486	.92747	1.40375	.99987	1.00000	.51075	.00004
.43060	.61831	-.30996	.12685	.94225	1.42612	.99987	1.01615	.46936	-.00004
.46225	.65036	-.26687	.12858	.95512	1.44559	.99987	1.03010	.40040	-.00003
.49400	.68246	-.22322	.13006	.96605	1.46215	.99987	1.04182	.33026	.00017
.52583	.71455	-.17914	.13127	.97503	1.47574	.99987	1.05131	.28768	-.00002

The exact values are:

$dV/dq_o = 0.0$

$H_{p_o} = 1.0$

$$V_x = U\left(1 + \frac{y^2 + z^2 - 2x^2}{2R^2}\right)$$

$$V_y = \frac{-1.5\ Uxy}{R^2} \qquad (44)$$

$$V_z = \frac{-1.5\ Uxz}{R^2}$$

where: U is the velocity of uniform flow at infinity

R is the radius of the shpere

Evidently, the streamlines of this flow are meridionals formed by planes passing through axis OX. The system of i and j lines was accepted as circles obtained through intersection of the sphere with two sets of planes perpendicular to plane XOY (see Figure 3). Results of the comparison of streamline computation for the case of 49 intersection planes in each direction is presented in Table 6. As seen from this table, the computations performed by the proposed method yield excellent accuracy. Table 7 shows results for computation of parameters in the final orthogonal curvilinear coordinate system p_o, q_o.

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